MAP2302 - Lecture Notes

Jeramiah Hocutt

June 11, 2017

Contents

1 Introduction

> Definition: A differential equation is an equation containing the derivatives of an unknown function.

Some examples you have most likely already seen:

- Free falling motion: $F = ma \Longrightarrow -mg = m\dfrac{d^2s}{dt^2}$
- Exponential decay: For $k > 0$, $\dfrac{dR}{dt} = -kR$
- Kirchhoff's Laws: $L\dfrac{d^2q}{dt^2} + R\dfrac{dq}{dt} + \dfrac{1}{C}q = E(t)$
- Wave equation: $\dfrac{\partial^2 u}{\partial t^2} - c^2\dfrac{\partial^2 u}{\partial x^2} = 0$ (This is a PDE)

These examples illustrate two types of differential equations. A differential equation with a single independent variable is called an ordinary differential equation (ODE). A differential equation with multiple independent variables is a called a partial differential equation (PDE). We will mostly restrict our attention to ODE's in this class.

The order of the differential equation is highest order derivative. So for example, in the free falling motion differential equation the order is 2.

> Definition: A linear differential equation is a differential equation that has only additive combinations of first powers of y and its derivatives.

That is, a linear differential equation has the form

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = F(x)$$

Some examples:

- $t^4\dfrac{dx}{dt} = t^2 + x$ is linear
- $\dfrac{d^2y}{dx^2} + y^2 = 0$ is nonlinear
- $\dfrac{d^2y}{dx^2} - y\dfrac{dy}{dx} = \cos(x)$ is nonlinear
- $\dfrac{d^2y}{dx^2} - x^2\dfrac{dy}{dx} + e^xy = \sin(x)$ is linear

Suggested HW: P.5 # 1-12, try # 13-16

Each differential equation can be written as

$$F\left(x, y, \frac{dy}{dx}, \ldots, \frac{d^ny}{dx^n}\right) = 0.$$

Or more often, when possible,

$$\frac{d^ny}{dx^n} = f\left(x, y, \frac{dy}{dx}, \ldots, \frac{d^{n-1}y}{dx^{-1}}\right).$$

Any $g(x)$ which can be substituted for y and satisfies this equation is called an explicit solution of the differential equation.

Ex: Show that each of the following are explicit solutions of the respective differential equation.

- $g(x) = x^2 - x^{-1}$ is a solution to $\dfrac{d^2y}{dx^2} - \dfrac{2}{x^2}y = 0$
- $g(x) = x^5$ is a solution to $\dfrac{dy}{dx} - \dfrac{5}{x}y = 0$
- $g(x) = \sin(2x)$ is a solution to $\dfrac{d^2y}{dx^2} + 4y = 0.$
- $g(x) = e^{4x}$ is a solution to $\dfrac{d^2y}{dx^2} - 16y = 0.$
- $g(x) = e^{x^2} + 3$ is a solution to $\dfrac{dy}{dx} - 2xy = -6x.$

Not all solutions are explicit solutions. Often, we have implicit solutions; that is, a solution of the form $g(x, y) = 0$.

Ex: Show that the relation $y^3 - \frac{3}{2}x^2 = 0$ is an implicit solution to $\dfrac{dy}{dx} = \dfrac{x}{y^2}$.

In the process of solving a differential equation we will come across constants from integration. The determination of these constants is one of the most important aspects of this course which we will be discussing regularly. Sometimes, there are solutions that are true regardless of the coefficients.

Ex: Show that for any constants c_1, c_2, the function $g(x) = c_1e^{-x} + c_2e^{2x}$ is an explicit solution to the linear equation $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 2y = 0.$

For many problems, we are given conditions on the desired solution function that allows us to solve for the constants of integration.

Definition: An initial value problem (IVP) is a solution to an n^{th} order differential equation $F\left(x, y, \dfrac{dy}{dx}, \dots, \dfrac{d^ny}{dx^n}\right) = 0$ on the interval I which satisfies the at $a \in I$, the initial conditions $y(a) = y_0, \dfrac{dy}{dx}(a) = y_1, \dots, \dfrac{d^{n-1}y}{dx^{n-1}}(a) = y_{n-1}$.

Ex: Show that $g(x) = \sin(x) - \cos(x)$ is a solution to the IVP $\dfrac{d^2y}{dx^2} + y = 0$, $y(0) = -1$, $\dfrac{dy}{dx}(0) = 1$.

We are always interested in finding a solutions to a differential equation. However, we should decide if a solution exists. This leads to one of the important theorems of the course regarding first order IVPs:

Theorem: (Existence and Uniqueness Theorem):

Consider the following IVP: $\dfrac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$. If f and $\dfrac{dy}{dx}$ are continuous functions in some rectangle $R = \{(x, y) |\ a < x < b,\ , c < y < d\}$ that contains the point (x_0, y_0), then the IVP has a uniques solution $g(x)$ in some interval $x_0 - \delta < x < x_0 + \delta$, where δ is a positive number.

Suggested HW: P.13 # 3-8, # 9-12, # 23-28; read section 1.3 try p.21 # 1-8

2 First Order differential equation

2.1 Separable Equations

Definition: A separable equation is a differential equation of the form

$$\frac{dy}{dx} = f(x, y) = g(x) \cdot h(y)$$

where $g(x)$ only depends on x and $h(y)$ only depends on y.

In this first example, the differential equation is separable, but the second is not.

Ex: $\frac{dy}{dx} = \frac{2x + xy}{y^2 + 1}$

Ex: $\frac{dy}{dx} = 1 + xy$

The process to solve such a differential equation is as follows:

Multiply both sides by $k(y) = \frac{1}{h(y)}$ ⚠.

Then multiply both sides by the differential dx. As of now we have

$$k(y)\frac{dy}{dx}dx = g(x)dx$$

By the chain rule $\frac{dy}{dx}dx = dy$, so we have

$$k(y)dy = g(x)dx$$

Now integrate both sides with respect to their appropriate variables. We will consolidate all constants of integration to the right hand side.

$$\int k(y)dy = \int g(x)dx \Longrightarrow K(y) = G(x) + C$$

The note for caution comes from the fact that there could be solutions for y which are constant so that $h(y) = 0$. These are separate possible solutions, which we will have to consider.

Ex: $\frac{dy}{dx} = \frac{x - 5}{y^2}$

Ex: $\frac{dy}{dx} = \frac{y - 1}{x + 3}$, $y(-1) = 0$

Ex: $\frac{dy}{dx} = \frac{6x^5 - 2x + 1}{\cos(y) + e^y}$

Ex: $\frac{dy}{dx} = 2xy - 6x$

Ex: $\frac{dy}{dx} = \frac{x}{y^2}$, $y(0) = 2$

Suggested HW: P.43 # 1-26, # 29, # 30, # 37-38

2.2 Linear Equations

Recall a first order linear differential equation is of the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = b(x)$$

where $a_1(x), a_0(x)$, and $b(x)$ only depend on x.

There are two main ways this could be solved, in general. If it happens that $a_0(x) \equiv 0$, then the equation reduces to

$$a_1(x)\frac{dy}{dx} = b(x) \Longrightarrow y(x) = \int \frac{b(x)}{a_1(x)}dx + C$$

assuming $a_1(x)$ is also not zero, of course. The case where $a_0(x)$ is not identically zero is less straightforward.

Suppose it were the case that $a_0(x) = a_1(x)'$. Then the left hand side of the equation is just:

$$a_1(x)\frac{dy}{dx} + a_1(x)'y = \frac{d}{dx}(a_1(x)y)$$

So then the equation would become

$$\frac{d}{dx}(a_1(x)y) = b(x) \Longrightarrow y(x) = \frac{1}{a_1(x)}\left(\int b(x)dx + C\right).$$

However nice this looks, the likelihood of this exact circumstance is low. However, by manipulating our original equation, and multiplying by a specially designed factor, we can manufacture this kind of scenario.

We begin by rewriting the differential equation into <u>standard form</u>

$$\frac{dy}{dx} + P(x)y = Q(x)$$

where $P(x) = \dfrac{a_0(x)}{a_1(x)}$ and $Q(x) = \dfrac{b(x)}{a_1(x)}$. (Notice it's not really relevant to care about the case where $a_1(x) = 0$ since then we wouldn't have an equation to solve)

We want to find a function $\mu(x)$ such that if we multiply the standard form by $\mu(x)$

$$\mu(x)\frac{dy}{dx} + \mu(x)P(x)y = \mu(x)Q(x)$$

the left hand side is actually equal to $\dfrac{d}{dx}(\mu(x)y)$: That is, we want

$$\mu(x)\frac{dy}{dx} + \mu(x)P(x)y = \mu(x)\frac{dy}{dx} + \mu'(x)y$$

This implies that $\dfrac{d\mu}{dx} = \mu(x)P(x)$, which is a separable differential equation.

$$\frac{1}{\mu}d\mu = P(x)dx \Longrightarrow \mu(x) = e^{\int P(x)dx}$$

Using this $\mu(x)$, we can now rewrite the standard form as we want and obtain

$$\frac{d}{dx}(\mu(x)y) = \mu(x)Q(x)$$

which has the general solution

$$y(x) = \frac{1}{\mu(x)}\left(\int \mu(x)Q(x)dx + C\right).$$

Ex: Find the general solution to $\dfrac{1}{x}\dfrac{dy}{dx} - \dfrac{2y}{x^2} = x\cos(x),\ x > 0$

Ex: Solve the IVP $\dfrac{dy}{dx} + 3xy = 5x,\ y(0) = 0$

Ex: Find the general solution to $\dfrac{dy}{dx} + \dfrac{2y}{x} = \cos(x),\ x > 0.$

Suggested HW: P.51 # 1-22, # 25, # 27, # 28, # 30, # 35

2.3 Exact Equations

Let us take a moment and consider the following scenario. Suppose we have a function $F(x, y)$ and we are looking at it's level curves $F(x, y) = C$ for some C. Then if we were to solve for tangent lines to these curves we would differentiate and solve for $\frac{dy}{dx}$. That is,

$$\frac{d}{dx}F(x,y) = \frac{d}{dx}C \Longrightarrow \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0 \qquad \left(\frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y}\right)$$

If we multiply through by dx, we obtain the differential form $\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = 0$, where we remember that $dy = \frac{dy}{dx}dx$. The expression $dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy$ is called a <u>total differential</u>. Setting dF equal to 0 would allow us to solve for the slopes of the level curves $F(x, y) = C$.

We rewrite this equation by defining $M(x, y) = \frac{\partial F}{\partial x}$ and $N(x, y) = \frac{\partial F}{\partial y}$ so that

$$M(x, y)dx + N(x, y)dy = 0$$

In fact, any first order differential equation can be rewritten in this form (recall the equation in parenthesis above).

Ex: Rewrite the differential equation $\frac{dy}{dx} = -\frac{2xy^2+1}{2x^2y}$ in differential form.

Ex: Rewrite the differential equation $\frac{dy}{dx} = -\frac{xe^{xy}}{ye^{xy}+1}$ in differential form.

<u>Definition</u>: The differential form $M(x, y)dx + N(x, y)dy$ is called <u>exact</u> in a rectangle R if there is a function $F(x, y)$ such that

$$\frac{\partial F}{\partial x}(x,y) = M(x,y) \quad \text{and} \quad \frac{\partial F}{\partial y}(x,y) = N(x,y)$$

for all (x, y) in R. If $M(x, y)dx + N(x, y)dy$ is an exact form we call the equation

$$M(x, y)dx + N(x, y)dy = 0$$

an <u>exact equation</u>.

It would be good to be able to test for exactness, and using some tools from multivariate calculus we can do so (recall Clairaut's theorem).

<u>Theorem</u>: Suppose the first partial derivatives of $M(x, y)$ and $N(x, y)$ are continuous in a rectangle R. Then $M(x, y)dx + N(x, y)dy = 0$ is an exact equation in R if and only if

$$\frac{\partial M}{\partial y}(x,y) = \frac{\partial N}{\partial x}(x,y)$$

holds for all (x, y) in R.

Ex: Show that the differential forms from the past two examples are in fact exact.

Now we need to discuss how to solve these exact equations. Suppose $M(x, y)dx + N(x, y)dy = 0$ is exact. Then we are supposing $\frac{\partial F}{\partial x} = M(x, y)$. We can integrate this equation with respect to x to obtain a 'guess' for our solution $F(x, y)$:

$$F(x,y) = \int M(x,y)\ dx + g(y)$$

where $g(y)$ is some portion which only depends on y. Now we also know that $\dfrac{\partial F}{\partial y} = N(x, y)$. We can then differentiate our 'guess' with respect to y and then equate it with $N(x, y)$ to solve for $g'(y)$:

$$N(x, y) = \frac{d}{dy}\left(\int M(x, y)\, dx\right) + g'(y)$$

Finally, integrating $g'(y)$ with respect to y gives us $g(y)$. Substituting this back into our 'guess' gives us the solution $F(x, y) = C$ when we collect all integration factors.

It should be noted that we could have started this instead with $\dfrac{\partial F}{\partial y} = N(x, y)$ first, and we would arrive at the same conclusion.

Ex: Solve $(2xy - \sec^2(x))dx + (x^2 + 2y)dy = 0$ (make sure to check for exactness).

Ex: Solve $(1 + e^x y + xe^x y)dx + (xe^x + 2)dy = 0$ (make sure to check for exactness).

Ex: Solve $(xe^{xy})dx + (ye^{xy} + 1)dy = 0$ (make sure to check for exactness).

Ex: Is $(x+3x^3 \sin(y))dx+(x^4 \cos(y))dy = 0$ exact? If not, can you figure out what to multiply through by to make it exact?[1] If you can, solve it.

Suggested HW: P.61 # 1-26, # 27-28, # 29

2.4 Special Integration Factors

Suppose we have a linear differential equation in standard form and we try to put it in its differential form:

$$\frac{dy}{dx} + P(x)y = Q(x) \Longrightarrow (P(x)y - Q(x))dx + dy = 0$$

A it is quick to see that this is not exact. It might be nice if we could multiply by something to make it exact. It turns out that if we multiply by the integrating factor $\mu(x)$, we then get

$$(\mu(x)P(x)y - \mu(x)Q(x))dx + \mu(x)dy = 0$$

By checking the condition on exactness, it requires that $\mu'(x) = \mu(x)P(x)$ which is guaranteed from our definition of $\mu(x)$.

So we can generalize integration factors. If

$$M(x, y)dx + N(x, y)dy = 0$$

is not exact but

$$\mu(x, y)M(x, y)dx + \mu(x, y)N(x, y)dy = 0$$

is exact, then $\mu(x, y)$ is an integration factor.

Ex: Check that $\mu(x, y) = xy^2$ is an integrating factor for $(2y - 6x)dx + (3x - 4x^2 y^{-1})dy = 0$, and solve ⚠.

Be wary, that we may have introduced extraneous solutions when multiplying by μ. We need to always check potential solutions when $\mu = 0$.

[1] try multiplying by x^{-1}

How do we decide what the integration factor is? Well, if we take this multiplied expression and 'test for exactness' we see that the necessary condition is

$$M\frac{\partial \mu}{\partial y} - N\frac{\partial \mu}{\partial x} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mu$$

This is a partial differential equation and far beyond the capabilities of this class to solve. So we will have to be satisfied with partial results.

If we knew that $\mu(x,y) = \mu(x)$, then the 'test for exactness' would yield

$$\frac{d\mu}{dx} = \left(\frac{\partial M/\partial y - \partial N/\partial x}{N}\right)\mu$$

If instead $\mu(x,y) = \mu(y)$, then the 'test for exactness' would yield

$$\frac{d\mu}{dy} = \left(\frac{\partial N/\partial x - \partial M/\partial y}{M}\right)\mu$$

If we are in either of these two cases, we can solve for our integration factor μ.

We can now collect all of our results on exact integration factors. Consider $Mdx+Ndy = 0$. If $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, then the equation is exact. Otherwise, check if

$$\frac{\partial M/\partial y - \partial N/\partial x}{N}$$

is a function of just x, or if

$$\frac{\partial N/\partial x - \partial M/\partial y}{M}$$

is a function of just y. Then the integration factors can be found by solving the appropriate 'test for exactness' equations.

Ex: Solve $(2x^2 + y)dx + (x^2y - x)dy = 0$.

Ex: Solve $(x + y)dx + (yx^2 + 2y^2x + x)dy = 0$.

Suggested HW: P.67 # 1-12

2.5 Substitutions

We are now going to look at solving certain types of first order differential equations via substitutions.

Definition: Consider $\dfrac{dy}{dx} = f(x,y)$. If $f(x,y)$ can be expressed as a function of $\left(\dfrac{y}{x}\right)$ alone, then we call the equation homogeneous.

If the differential equation is homogeneous, we can perform the substitution $v = \dfrac{y}{x}$.

This makes $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$, and we can now solve using previous techniques.

A quick way to test for homogeneity is to check if $f(tx,ty) = f(x,y)$ for all $t \neq 0$.

Ex: Solve $(x - y)dx + xdy = 0$

Ex: Solve $(y^2 + xy)dx - x^2dy = 0$

Ex: Solve $\frac{dy}{dx} = \frac{y}{x} e^{\frac{y}{x}}$

Suggested HW: P.74 # 9-16

Definition: A first order equation which can be written as

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \; n \, \text{real}$$

where $P(x)$ and $Q(x)$ are continuous on an interval I is called a Bernoulli equation.

When solving equations of this form, we use the substitution $v = y^{1-n}$. This create a linear equation we can then solve. Observe, if we divide both sides by y^n, we get

$$y^{-n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x)$$

With our substitution we see that

$$v = y^{1-n} \Longrightarrow \frac{dv}{dx} = (1-n)y^{-n}\frac{dy}{dx}$$

Replacing $\frac{dy}{dx}$ and y^{1-n} we get a new linear equation

$$\frac{1}{1-n}\frac{dv}{dx} + P(x)v = Q(x)$$

which we can solve as normal.

Ex: Solve $\frac{dy}{dx} - 5y = -\frac{5}{2}xy^3$

Ex: Solve $\frac{dy}{dx} - y = e^x y^2$

Suggested HW: P.74 # 21-28

3 Some Real World Examples

3.1 Compartmental Type Examples

Many real-world systems can be viewed as what is known as a compartmental system (maybe black-box system). Where a quantity is fed into a system at a certain rate and then the quantity flows out of a system at another rate. Solving the associated differential equation gives insight into how the system affects the described quantity.

This idea is captured in the following, for a quantity x changing with respect to time t,

$$\frac{dx}{dt} = (\text{rate in}) - (\text{rate out}).$$

Ex: The first example we see is the typical mixing problem. For example, suppose a tank holds 1000 L of pure water solution and a brine solution of salt is being poured in at a constant rate of 6 L/min. The solution in the tank is well-stirred, and flows out of the tank at a rate of 6 L/min. If the concentration of salt in the brine entering the tank is 0.1kg/L, we want to know when the concentration of salt in the tank will reach 0.05kg/L.

Since we are concerned with the concentration of salt, let us define $x(t)$ to be the mass of salt in the tank at any given time t. To get the concentration, we will need to divide $x(t)$ by the volume of liquid in the tank (which is always 1000L since the rate of liquid flow is the same in and out).

To set up the differential equation, we need to know the rate in and rate out of salt. The brine flows in at 6L/min and the concentration of salt entering is $0.1kg/L$, so the overall rate of salt flow is 0.6kg/min. The rate out of the tank depends on the concentration in the tank (which is uniform since it is well-stirred). So the rate out would be (6L/min) $*$ ($x(t)$/1000kg/L) = $3x(t)$/500kg/min.

So our differential equation is then

$$\frac{dx}{dt} = 0.6 - \frac{3x}{500}, \qquad x(0) = 0$$

where $x(0) = 0$ since there is no salt initially in the tank. We can now solve this using our regular first order methods.

Try deciding how the volume and concentrations change when the rate leaving the tank is 5 L/min instead.

Ex: Another example of these compartment type problems is that of population growth. Typical population growth is proportional to the current population. In this way, we have that the appropriate differential equation is

$$\frac{dp}{dt} = k \cdot p, \qquad p(0) = P_0$$

where we assume that the initial population was P_0. We can also include death rates (assuming natural causes) and we arrive at effectively the same equation:

$$\frac{dp}{dt} = k_b \cdot P - k_d \cdot P = K \cdot p, \qquad p(0) = P_0$$

The solution to this is a simple exponential model. We can add one more type of rate effect, that of two-person interaction (think predator/prey). The ways these interactions can take place is $p(p-1)/2$ possible ways. So we now have

$$\frac{dp}{dt} = k_1 - k_2\frac{p(p-1)}{2} = -Ap(p-B), \qquad p(0) = P_0$$

where $A = k_2/2$, and $B = (2k_1/k_2) + 1$. We can solve this separable equation and obtain the following:

$$p(t) = \frac{B}{1-(1-B/P_0)e^{-ABt}}$$

This function is called the logistic function. Observe the behavior as $t \to \infty$ and as $t \to -\infty$. These asymptotes ($y = 0$ and $y = B$) reveal what are known as equilibrium solutions (i.e. constant solutions). These represent the extreme ends of the logistic model.

Ex: Suppose in 1790 the population was 3.93 million people, and assume that the 1840 and 1890 populations were 17.07 and 62.98 million respectively. Estimate the growth of the population.

Suggested HW: P.99 # 1-9, # 14, # 21-22

3.2 Circuit Examples

We will now consider RC and RL circuits. The voltage drop across a resistor of resistance R is $E_R = RI$, where I is the current in the circuit. The voltage drop across an inductor of inductance L is given by $E_L = L\frac{dI}{dt}$. The voltage drop across a capacitor of capacitance C is given by $E_C = \frac{1}{C}q$ where q is the charge on the capacitor.

Using Kirchoff's voltage law, we can determine the differential equation for the current in an RL circuit with voltage source $E(t)$:

$$E_L + E_R = E(t) \Longrightarrow L\frac{dI}{dt} + RI = E(t)$$

This is a first order linear differential equation which we can solve using the techniques we know:

$$I(t) = e^{Rt/L}\left(\int e^{Rt/L}\frac{E(t)}{L}dt + K\right).$$

Ex: An RL circuit with a 1-Ω resistor and a $0.01-$H inductor is driven by a voltage $E(t) = \sin(100t)$ V. If the initial current is 0, determine the resistor and inductor voltages and the current.

Using Kirchoff's voltage law again, we can determine the differential equation for the charge q:

$$E_R + E_C = E(t) \Longrightarrow RI + \frac{1}{C}q = E(t) \Longrightarrow R\frac{dq}{dt} + \frac{1}{C}q = E(t)$$

Again, this is a first order linear equation and can be solved with the techniques we know.

Ex: Suppose a capacitor of C farads holds an initial charge Q coulombs. A constant voltage V volt is applied through a resistor of R ohms. What is the capacitor charge for $t > 0$.

Note if $V = 0$ in this example, we obtain the time-constant which describes how long it takes for the capacitor to lose charge.

Suggested HW: P.121 # 1-2

4 Second Order Equations

4.1 Homogeneous Linear Equations

We begin looking at linear second order differential equations with constant coefficients

$$a\ddot{y} + b\dot{y} + cy = f(t), (a \neq 0)$$

and we consider the case when $f(t) = 0$:

$$a\ddot{y} + b\dot{y} + cy = f(t).$$

This is called the homogeneous form of the equation. If we look at the equation, we see that it necessarily must have the property that it's second derivative is a linear combination of it's first and zeroth derivative. A good guess at the solution might be an exponential solution e^{rt}.

We can substitute this into our equation and obtain:

$$ar^2e^{rt} + bre^{rt} + ce^{rt} = 0 \Longrightarrow e^{rt}(ar^2 + br + c) = 0$$

Since exponentials are nonzero, this leads to $ar^2 + br + c = 0$. This equation must be satisfied for e^{rt} to be a solutions. This equation is called the characteristic equation (or auxiliary equation) associated with the homogeneous differential equation. This characteristic equation has two roots given by the quadratic formula

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

When the discriminant is positive, the two roots r_1, r_2 are distinct real numbers; zero discriminant gives two, equal real roots; and a negative discriminant gives two, complex conjugate roots. We will first look at the case of a non-negative discriminant.

Ex: Find a pair of solutions to $\ddot{y} + 5\dot{y} - 6y = 0$.

Ex: Find a pair of solutions to $\ddot{y} + 7\dot{y} + 12y = 0$.

Ex: Find a pair of solutions to $2\ddot{y} - \dot{y} - y = 0$.

Notice that the examples have been asking for a pair of solutions, not all solutions. This is because the two you have found do not constitute all the possible solutions. In fact, given a pair of solutions, y_1, y_2, we can create an infinite number of solutions by linear combinations. That is, a general solution would be

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

for some choice of constants c_1, c_2. These two degrees of freedom, as in the constants of integration from earlier, represent the parameters for solutions. We would say that the solutions set is a 2-parameter family of solutions.

An IVP would require two pieces of initial data to be able to determine a precise solutions, since we have two unknown parameters.

Ex: Solve the IVP $\ddot{y} + 2\dot{y} - y = 0, \quad y(0) = 0, \, \dot{y}(0) = -1$.

Ex: Solve the IVP $\ddot{y} + 3\dot{y} - 10y = 0, \quad y(0) = 2, \, \dot{y}(0) = 4$.

There is an existence and uniqueness theorem for the homogeneous case of second order linear equations:

Theorem: For any real numbers $a \neq 0, b, c, t_0, y_0, y_1$, there exists a unique solution to the IVP

$$a\ddot{y} + b\dot{y} + cy = 0; \quad y(t_0) = y_0, \; \dot{y}(t_0) = y_1$$

and this solution is valid for all t in $(-\infty, \infty)$.

Typically, the difficult part is showing uniqueness of solutions, as opposed to existence. Given two solutions $y_1(t), y_2(t)$ to the differential equation, we can find constants c_1, c_2 so that we have a unique solution $c_1 y_1(t) + c_2 y_2(t)$ to the two conditions given in the IVP. However, what if one of the solutions, say, y_2 was identically 0? Then the solution would actually be just $c_1 y_1(t)$, and this cannot satisfy the *two* condition IVP since it only has one constant.

What if y_2 was just a multiple of y_1? That is, $y_2(t) = k y_1(t)$. This would imply that again we only have a one parameter family (i.e. $c_1 y_1(t) + c_2 y_2(t) = (c_1 + c_2 k) y_1(t) = C y_1(t)$). This still cannot satisfy an IVP with two initial conditions. We need a condition on the solutions to guarantee a two parameter family of solutions in order to be able to satisfy the IVP. The condition is one of the most important mathematical concepts discussed in this class; that of *linear independence*.

Definition: A pair of functions $y_1(t), y_2(t)$ is said to be linearly independent (on the interval I) if and only if neither is a constant multiple of the other. If one is a constant multiple of the other, then we say that the pair is linearly dependent.

Theorem: If $y_1(t), y_2(t)$ are any two solutions to the second order linear homogeneous differential equation that are linearly independent, then unique constants c_1, c_2 can always be found so that $c_1y_1(t)+c_2y_2(t)$ satisfies a given IVP on $(-\infty, \infty)$.

A condition to check for linear dependence is as follows:

For any two solutions $y_1(t), y_2(t)$ to the second order linear homogeneous differential equation, if the equality

$$y_1(t)\dot{y_2}(t) - \dot{y_1}(t)y_2(t) = 0$$

holds at any point t, then y_1 and y_2 are linearly dependent on $(-\infty, \infty)$.

The expression $y_1(t)\dot{y_2}(t) - \dot{y_1}(t)y_2(t)$ is called the Wronskian of y_1 and y_2 at the point t. Typically denoted, $W[y_1, y_2]$, the Wronskian, in the case of these second order solutions, is the determinant of the matrix

$$W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ \dot{y_1} & \dot{y_2} \end{vmatrix} = y_1(t)\dot{y_2}(t) - \dot{y_1}(t)y_2(t).$$

Now that we have this understanding, we can go back to the solutions we were finding using the characteristic equation. Notice that when the roots were distinct and real, the solutions $y_1(t) = e^{r_1t}$ and $y_2(t) = e^{r_2t}$ are automatically linearly independent since $r_1 \neq r_2$. This means that in this case the general solution is indeed

$$y(t) = c_1e^{r_1t} + c_2e^{r_2t}.$$

If the roots are real but repeated, then both $y_1(t) = e^{rt}$ and $y_2(t) = te^{rt}$ are solutions with the general solution

$$y(t) = c_1e^{rt} + c_2te^{rt}.$$

The reason $y_2(t) = te^{rt}$ is also a solution is not immediately apparent. We will discuss this fact in general later on, but for now try taking te^{rt} and substituting it into the differential equation to see it is in fact a solution.

Ex: Solve the IVP $\ddot{y} + 4\dot{y} + 4y = 0; \quad y(0) = 1,\ \dot{y}(0) = 3.$

Ex: Find the general solution to $\ddot{y} + 8\dot{y} + 16y = 0.$

Ex: Find the general solution to $\dddot{y} + 3\ddot{y} - \dot{y} - 3y = 0$

Suggested HW: P.165 # 1-20, # 27-32, # 35 (a-c, try d if you know cosh**)**

We now need to talk about the case where the discriminant of the characteristic equation is negative. In this case, the roots of the characteristic equation are complex conjugates, so

$$r_1 = \alpha + \beta i \quad \text{and} \quad r_2 = \alpha - \beta i.$$

We want to propose that solutions would have the form e^{r_1t} and e^{r_2t} as before. However, we need to ask what we mean with $e^{(\alpha+\beta i)t}$. It is a fact that the laws of exponents holds for complex numbers. This means that $e^{(\alpha+\beta i)t} = e^{\alpha t}e^{\beta it}$. To describe $e^{\beta it}$, we should perhaps examine a Taylor series expansion for the generic $e^{i\theta}$ (assuming it exists and is the same for complex numbers). Computing the Taylor series and collecting the appropriate terms, we have the following identity (known as Euler's formula)

$$e^{i\theta} = \cos(\theta) + i\sin(\theta).$$

Using this, we see that a solution of the form $e^{(\alpha+\beta i)t}$ can be written as

$$e^{(\alpha+\beta i)t} = e^{\alpha t}(\cos(\beta t) + i\sin(\beta t))$$

and similarly for its complex conjugate.

This means a general solution would have the form

$$y(t) = c_1 e^{\alpha t}(\cos(\beta t) + i\sin(\beta t)) + c_2 e^{\alpha t}(\cos(\beta t) - i\sin(\beta t)).$$

Ex: Solve the IVP $\ddot{y} + 2\dot{y} + 2y = 0$, $y(0) = 0$, $\dot{y}(0) = 2$.
Observe that these general solutions require complex arithmetic. We would like to remain in the real realm if possible. It is a fact of complex functions that if $z(t)$ is a complex function, then we can write $z(t) = u(t) + iv(t)$, where $u(t)$ and $v(t)$ are real-valued functions.

If $z(t)$ is a solution to the homogeneous second order linear differential equation , then $u(t)$ and $v(t)$ are real-valued solutions.

If the characteristic equation has complex conjugate roots $\alpha \pm \beta i$, then two linearly independent solutions are $e^{\alpha t}\cos(\beta t)$ and $e^{\alpha t}\sin(\beta t)$, and a general solutions is $c_1 e^{\alpha t}\cos(\beta t) + c_2 e^{\alpha t}\sin(\beta t)$ for arbitrary coefficients c_1, c_2.

Ex: Find a general solution to $\ddot{y} + 2\dot{y} + 4y = 0$.

Suggested HW: P.173 # 1-27

4.2 Nonhomogeneous Equations

We desire now to develop methods to solve the nonhomogeneous linear second order equations. We begin by describing an important fact regarding solutions to differential equations.

Theorem: (Superposition Principle):
Let y_1 be a solution to $a\ddot{y} + b\dot{y} + cy = f_1(t)$, and y_2 be a solution to $a\ddot{y} + b\dot{y} + cy = f_2(t)$. Then for any constants k_1, k_2, the function $k_1 y_1 + k_2 y_2$ is a solution to the differential equation $a\ddot{y} + b\dot{y} + cy = k_1 f_1(t) + k_2 f_2(t)$.

Using the superposition principle, we can arrive at a way of computing the general solution to a nonhomogeneous equation. Given a second order linear nonhomogeneous differential equation

$$a\ddot{y} + b\dot{y} + cy = f(t)$$

to find a general solution, we first find a general solution to the homogeneous equation $a\ddot{y} + b\dot{y} + cy = 0$, call it y_h, and then we find a particular solution to the nonnhomogeneous equation $a\ddot{y} + b\dot{y} + cy = f(t)$, call it y_p. Then a general solution to the nonhmogeneous equation would be, by the superposition principle, the sum

$$y(t) = y_h + y_p.$$

So our goal is to be able to find particular solutions to the nonhomogeneous equation. There are a number of techniques for doing this. We will focus on two: undetermined coefficients and variation of parameters.

4.2.1 Undetermined Coefficients

We will proceed by looking at various examples. First consider the nonhomogeneous equation

$$\ddot{y} + 3\dot{y} + 2y = 3t$$

We need to find a particular solution y_p such that a combination of it with its first and second derivatives gives the linear function $3t$. A guess might be another linear function $y_p = At$, for some constant A. However, after trying this in the equation, we see that it is impossible for this to work. Perhaps we did not use a proper linear function. Let's try $y_p(t) = At + B$.

If we substitute in the first and second derivatives we get

$$2At + (3A + 2B) = 3t$$

which we can solve explicitly for A and B - namely $A = \frac{3}{2}$, $B = -\frac{9}{4}$.

A quick look at the differential equation $\ddot{y} + 3\dot{y} + 2y = 3t$ suggests we might try the particular solution $y_p = At^2 + Bt + C$. Indeed, substituting this into the equation does present a solvable system. It turns out we can refine this to a more general case. To find a particular solution to the differential equation

$$a\ddot{y} + b\dot{y} + cy = Kt^n$$

we consider the solution given by

$$y_p(t) = A_n t^n + A_{n-1} t^{n-1} + \cdots + A_1 t + A_0$$

Now let us consider the the nonhomogeneous equation

$$\ddot{y} + 3\dot{y} + 2y = 10e^{3t}.$$

A guess for the particular solution would be $y_p(t) = Ae^{3t}$. Substituting this into the differential equation we easily see that this gives a solution with $A = \frac{1}{2}$.

This gives a general case for the differential equation

$$a\ddot{y} + b\dot{y} + cy = Ke^{rt}$$

by using the particular solution given by

$$y_p(t) = Ae^{rt}.$$

Now we come to another type of nonhomogeneous equation:

$$\ddot{y} + 3\dot{y} + 2y = \sin(t)$$

We might try the particular solution $y_p(t) = A\sin(t)$. However, notice that when we take derivatives, we will have a $\cos(t)$ term that will arise in the substituted equation that we cannot account for. So this leads us to amend out guess for the particular solution to

$$y_p(t) = A\sin(t) + B\cos(t).$$

When we substitute this into our differential equation we arrive at the system of equations

$$\begin{array}{rcl}(A - 3B) & = & 1 \\ (B + 3A) & = & 0\end{array}$$

Which gives $A = \frac{1}{10}$, $B = -\frac{3}{10}$.
The general result for this type being for the differential equation

$$a\ddot{y} + b\dot{y} + cy = K\sin(\beta t) \quad \text{or} \quad a\ddot{y} + b\dot{y} + cy = K\cos(\beta t)$$

the desired particular solution has the form

$$y_p(t) = A\sin(\beta t) + B\cos(\beta t).$$

We would like to understand what happens when our forcing function has a form that combines some of these past examples, e.g. $\ddot{y} + 4y = 5t^2e^t$. We might simply try multiplying the associated particular solutions, and indeed this works most of the time. However, there are some unfortunate cases.

For example, the differential equation $\ddot{y} + \dot{y} = 5$, or $\ddot{y} - 6\dot{y} + 9y = e^{3t}$. The issues for these arise from the fact that the guesses for our particular solutions are actually already found in the homogeneous solutions. For the first equation, observe that one of the roots of the characteristic equation is $r = 0$. So we would have $c_1e^{0t} = c_1$ as part of the homogeneous solution, so we can't simply guess $y_p = A$ since it was already accounted for in y_h. We would have to play the trick of multiplying by t to bypass the issue, i.e. try $y_p = At$.

Similarly, for the second equation, $r = 3$ is a double root of the characteristic equation so guessing $y_p = Ae^{3t}$ doesn't work. Moreover, if we try to bypass it like before by multiplying by an extra t, $y_p = Ate^{3t}$, this also doesn't work since this too is present in the homogeneous solution. We would need to bypass the bypass, as it were, and try $y_p = At^2e^{3t}$. For both of these, the number of t's we need to multiply depended on whether or not our guess for the particular solution was already accounted for in the homogeneous solution. We summarize the results below.

- For $a\ddot{y} + b\dot{y} + cy = Kt^n$, use $y_p(t) = A_nt^n + A_{n-1}t^{n-1} + \cdots + A_1t + A_0$
- For $a\ddot{y} + b\dot{y} + cy = Ke^{rt}$, use $y_p(t) = Ae^{rt}$
- For $a\ddot{y} + b\dot{y} + cy = K\sin(\beta t)$ or $a\ddot{y} + b\dot{y} + cy = K\cos(\beta t)$, use $y_p(t) = A\sin(\beta t) + B\cos(\beta t)$.
- For $a\ddot{y} + b\dot{y} + cy = Kt^ne^{rt}$, use $y_p(t) = t^s(A_nt^n + \cdots + A_1t + A_0)e^{rt}$ where
  - $s = 0$ if r is not a root of the characteristic equation
  - $s = 1$ if r is a simple root of the characteristic equation
  - $s = 2$ if r is a double root of the characteristic equation
- For $a\ddot{y} + b\dot{y} + cy = Kt^ne^{\alpha t}\cos(\beta t)$ or $a\ddot{y} + b\dot{y} + cy = Kt^ne^{\alpha t}\sin(\beta t)$ use

  $$y_p(t) = t^s(A_nt^n + \cdots + A_1t + A_0)e^{\alpha t}\sin(\beta t) + t^s(B_nt^n + \cdots + B_1t + B_0)e^{\alpha t}\cos(\beta t)$$

  where
  - $s = 0$ if $\alpha + \beta i$ is not a root of the characteristic equation
  - $s = 1$ if $\alpha + \beta i$ is a root of the characteristic equation

Ex: What is the appropriate form for the particular solution for the following nonhomogeneous equations?

- $\ddot{y} + 2\dot{y} - 3y = 7\cos(3t)$
- $\ddot{y} + 2\dot{y} - 3y = 2te^t\sin(t)$
- $\ddot{y} + 2\dot{y} - 3y = t^2\cos(\pi t)$
- $\ddot{y} + 2\dot{y} - 3y = 5e^{-3t}$
- $\ddot{y} + 2\dot{y} - 3y = 3te^t$
- $\ddot{y} + 2\dot{y} - 3y = t^2e^t$

Ex: Find the appropriate form for the particular solution for the nonhomogeneous equation $\ddot{y}-2\dot{y}+y = f(t)$, where you replace $f(t)$ with each of the forcing functions from the previous question.

Suggested HW: P.182 # 9-32

Suggested HW: P.187 # 3-8, # 17-22, # 23-26, # 31-36

4.2.2 Variation of Parameters

Consider the nonhomogeneous equation $a\ddot{y} + b\dot{y} + cy = f(t)$, and suppose $y_1(t), y_2(t)$ is a pair of linearly independent solutions for the homogeneous equation $a\ddot{y} + b\dot{y} + cy = 0$. Normally, for a general solution to the homogeneous equation we would take $y_h(t) = c_1y_1(t) + c_2y_2(t)$. To find a particular solution to the nonhomogeneous equation, we allow our coefficients to be functions of t:

$$y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t).$$

If this is to be a solution to $a\ddot{y} + b\dot{y} + cy = f(t)$, then we can substitute our particular solution and use it to solve for the new variable coefficients. First we compute $\dot{y}_p$

$$\dot{y}_p = (\dot{v}_1y_1 + \dot{v}_2y_2) + (v_1\dot{y}_1 + v_2\dot{y}_2)$$

To make computations easier, we impose a condition that $\dot{v}_1y_1 + \dot{v}_2y_2 = 0$. This will result in one of our two constraining equations. This restriction means that $\dot{y}_p = v_1\dot{y}_1 + v_2\dot{y}_2$. Taking second derivatives, we now have

$$\ddot{y}_p = \dot{v}_1\dot{y}_1 + v_1\ddot{y}_1 + \dot{v}_2\dot{y}_2 + v_2\ddot{y}_2.$$

If we substitute this into $a\ddot{y}+b\dot{y}+cy = f(t)$ and simplify. Because y_1, y_2 are solutions to the homogeneous equation, the simplifies result becomes

$$f(t) = a(\dot{v}_1\dot{y}_1 + \dot{v}_2\dot{y}_2) \Longrightarrow \dot{v}_1\dot{y}_1 + \dot{v}_2\dot{y}_2 = \frac{f}{a}.$$

This gives the second constraint on $\dot{v}_1, \dot{v}_2$. So we need only solve the following system for $\dot{v}_1, \dot{v}_2$:

$$\begin{aligned} \dot{v}_1y_1 + \dot{v}_2y_2 &= 0 \\ \dot{v}_1\dot{y}_1 + \dot{v}_2\dot{y}_2 &= \tfrac{f}{a} \end{aligned}$$

Once we have expressions for $\dot{v}_1, \dot{v}_2$, we can integrate to find v_1, v_2. This will give us a particular solution, which we add to the homogeneous general solution to get a general solution to the nonhomogeneous differential equation .

Ex: Find a general solution on $(-\pi/2, \pi/2)$ to $\ddot{y} + y = \tan(t)$.

Ex: Find a particular solution on $(-\pi/2, \pi/2)$ to $\ddot{y}+y = \tan(t)+3t-1$. (Hint: use superposition and undetermined coefficients)

Ex: Find a particular solution to $\ddot{y} + 6\dot{y} + 9y = \dfrac{e^{-3t}}{t^2}$.

Ex: Find a particular solution to $\ddot{y} - 4\dot{y} + 4y = e^{2t}\sqrt{t}$.

Suggested HW: P.193 # 1-18, read section 4.9, 4.10

5 Higher Order Linear Differential Equations

5.1 Differential Operators

Recall that a linear differential equation of order n has the form

$$a_n(x)\frac{d^n y}{dx^n} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = b(x)$$

If we divide by $a_n(x)$ we can rewrite this equation into standard form:

$$\frac{d^n y}{dx^n} + p_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + p_{n-1}(x)\frac{dy}{dx} + p_n(x)y(x) = q(x)$$

where the functions $p_1(x), \ldots, p_n(x), q(x)$ are continuous on some interval I. There is an existence and uniqueness theorem for solutions to initial value problems of linear higher order differential equations. (See in textbook).

For our purposes in moving forward, we will define the differential operator L

$$L[y] := \frac{d^n y}{dx^n} + p_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + p_{n-1}\frac{dy}{dx} + p_n y = (D^n + p_1 D^{n-1} + \cdots + p_n)[y].$$

With this definition in hand, we rewrite the standard form differential equation as

$$L[y](x) = g(x).$$

This L is a linear operator, meaning that for $y_1, \ldots, y_n$

$$L[y_1 + y_2 + \cdots + y_n] = L[y_1] + L[y_2] + \cdots + L[y_n]$$

and for c a constant

$$L[cy] = cL[y].$$

Now suppose that $y_1, \ldots, y_n$ are solutions to the homogeneous equation $L[y](x) = 0$. By the above linearity fact, any linear combination of these solutions (i.e. $C_1y_1 + \cdots + C_ny_n$) will also be a solution since

$$L[C_1y_1 + C_2y_2 \cdots + C_ny_n] = C_1L[y_1] + C_2L[y_2] + \cdots + C_nL[y_n] = C_1 \cdot 0 + \cdots + C_n \cdot 0 = 0.$$

We now extend the fact we derived back in the second order case. If $y_1, \ldots, y_n$ on (a, b) are solutions to

$$\frac{d^n y}{dx^n} + p_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + p_{n-1}\frac{dy}{dx} + p_n y = 0$$

where each p_i is continuous on (a, b) and these solutions form a linearly independent set, then every solution of the given differential equation can be expressed as

$$y(x) = C_1y_1(x) + \cdots + C_ny_n(x)$$

where $C_1, \ldots, C_n$ are constants.

Recall that being linearly independent means that none of the solutions can be written as a linear combination of the others. That is, if $C_1y_1 + \cdots + C_ny_n = 0$, then it must be the case that all of the $C_1, \ldots, C_n$ are 0. Alternatively, one could instead say that the set is linearly dependent if one solution can be written as a linear combination of the others. That is, there exist $C_1, \ldots, C_n$ not all 0, such that $C_1y_1 + \cdots + C_ny_n = 0$.

The condition that the solutions form a linearly independent set can be verified by computing the Wronskian and showing it is nonzero at some point in (a, b). The Wronskian is defined as

$$W[y_1, \ldots, y_n](x) := \det \begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix}$$

Suggested HW: P.325 # 7-12, # 15-17, # 19-22, # 23, # 24

5.2 Homogeneous Linear Equations

Consider the homogeneous linear n^th order differential equation

$$a_n \frac{d^n y}{dx^n} + \cdots + a_1 \frac{dy}{dx} + a_0 y = 0.$$

We know that a general solution to this would have the form $y(x) = C_1 y_1 + \cdots + C_n y_n$. If we can find n linearly independent solutions, we will be able to construct this. We might try to see how things work if we begin as we did in the second order case, i.e. try $y = e^{rx}$. If we substitute this into our homogeneous equation we see that

$$a_n r^n e^{rx} + a_{n-1} r^{n-1} e^{rx} + \cdots + a_0 e^{rx} = e^{rx}(a_n r^n + a_{n-1} r^{n-1} + \cdots + a_0)$$

We define $P(r) = a_n r^n + a_{n-1} r^{n-1} + \cdots + a_0$ to be the characteristic polynomial. Again, as before, we see that e^{rx} is a solution to the differential equation as long as r is a root of characteristic equation $P(r) = 0$. We will have to consider cases in this setting as well. We begin with distinct real roots.

If $r_1, \ldots, r_n$ are n distinct real roots of the characteristic equation, then n solutions to the homogeneous differential equation are

$$y_1(x) = e^{r_1 x}, y_2(x) = e^{r_2 x}, \ldots, y_n(x) = e^{r_n x}$$

These solutions are can be verified to be linearly independent (by computing the Wronskian, for example). Therefore, a general solution to the homogeneous equation is

$$y(x) = C_1 e^{r_1 x} + \cdots + C_n e^{r_n x}$$

where $C_1, \ldots, C_n$ are arbitrary constants.

Ex: Find a general solution to $y''' - 2y'' - 5y' + 6y = 0$. (Hint: try $r = 1$)

We now examine the case of complex roots. If $\alpha + \beta i$ is a root of the characteristic polynomial (and hence so is $\alpha - \beta i$), then two linearly independent solutions would be $e^{\alpha x}\cos(\beta x)$ and $e^{\alpha x}\sin(\beta x)$. Since complex roots come in conjugate pairs, we have that these will be the two solutions representing these roots in the previous general solution.

Ex: Find a general solution to $y''' + y'' + 3y' - 5y = 0$. (Hint: try $r = 1$)

Finally, we tackle the case of repeated roots. If r_1 is a root with multiplicity m, then in order to make n linearly independent solutions, we multiply each copy by an appropriately increasing power

$$e^{r_1 x}, xe^{r_1 x}, x^2 e^{r_1 x}, \ldots, x^{m-1} e^{r_1 x}.$$

If the repeated root is complex, then since they come in conjugate pairs, we have to make this amendment to each set of solutions (i.e. both the cos and sin solutions).

Ex: Find a general solution to $y^{(4)} - y^{(3)} - 3y'' + 5y' - 2y = 0$ (Hint: try $r = 1$).

Ex: Find a general solution to $y^{(4)} - 8y^{(3)} + 26y'' - 40y' + 25y = 0$. (Hint: $P(r)$ corresponds to $(r^2 - 4r + 5)^2$)

Suggested HW: P.331 # 1-14, # 19-21

5.3 Annihilator Method

In this section we will generalize the notions of undetermined coefficients that were used to solve second order equations. To begin with, we will recall what we described for second order equations in terms of differential operators. For example, if we consider the differential operator

$$L[y] = (aD^2 + bD + c)[y] = f(x)$$

this is the nonhomogenous second order linear differential equation that we are familiar with. We know what particular solutions would look like depending on type of equation:

- $(D - r)[f] = 0$ holds when $f = e^{rx}$
- $(D - r)^m[f] = 0$ holds when $f = x^k e^{rx}$ for $k = 0, 1, \ldots, m - 1$
- $(D^2 + \beta^2)[f] = 0$ holds when $f = \cos(\beta x)$ or $f = \sin(\beta x)$
- $[(D - \alpha)^2 + \beta^2]^m[f] = 0$ holds when $f = x^k e^{\alpha x} \cos(\beta x)$ or $f = x^k e^{\alpha x} \sin(\beta x)$ for $k = 0, 1, \ldots, m - 1$

These forcing functions are annihilated by the differential operators.

Definition: A linear differential operator A annihilates a function f if $A[f](x) = 0$ for all x. Or in other words, A annihilates f if f is a solution to the above homogeneous linear differential equation.

Ex: What differential operator annihilates $6xe^{-4x} + 5e^x \sin(2x)$?

How do annihilators work? Given the n^{th} order nonhomogeneous differential equation written in operator form $L[y](x) = f(x)$, we assume that there is a linear differential operator A, which annihilates $f(x)$. That is $A[L[y]](x) = A[f](x) = 0$. This means that any solution to the original differential equation must be a solution to the homogeneous differential equation $AL[y](x) = 0$.

As an example, find a general solution to $y'' - y = xe^x + \sin(x)$. We could solve this with old techniques, but let's try the new ones. Observe that $(D^2 + 1)$ annihilates $\sin(x)$, and $(D - 1)^2$ annihilates xe^x. The original differential equation can be expressed as

$$(D^2 - 1)[y](x) = xe^x + \sin(x)$$

When we compose our operators we see that it is sufficient to find solutions that satisfy the homogeneous differential equation

$$(D^2 + 1)(D - 1)^2(D^2 - 1)[y] = (D + 1)(D - 1)^3(D^2 + 1)[y] = 0$$

Notice from the last section that a particular solution would have the form $Axe^x + Bx^2e^x + C\sin(x) + D\cos(x)$.

Ex: Find a general solution to $y''' - 3y'' + 4y = xe^{2x}$

Suggested HW: P.337 # 1-30

6 The Laplace Transform

6.1 Definition of the Laplace transform

We begin this section by defining a new type of operator that transforms functions of one variable into functions of a different variable. The purpose being to take functions and convert them to new functions with nicer properties to work with. We begin by defining the **Laplace Transform**.

Definition: Let $f(t)$ be a function on $[0,\infty)$. The Laplace Transform of f is the function $F(s)$ defined by the integral

$$\mathscr{L}\{f\}(s) = F(s) := \int_0^\infty e^{-st} f(t)\, dt \qquad \left(= \lim_{N\to\infty} \int_0^N e^{-st} f(t)\, dt\right)$$

The domain of $F(s)$ is all the values of s for which the integral exists.

We now begin by looking at some examples of Laplace transforms of simple, common functions:

Ex: Determine the Laplace transform of $f(t) = 1$, $t \geq 1$.

$$\mathscr{L}\{1\} = \int_0^\infty e^{-st}(1)\, dt = \lim_{N\to\infty} \int_0^N e^{-st}\, dt = \lim_{N\to\infty} \left(\frac{-e^{-st}}{s}\right)\Bigg|_0^N = \frac{1}{s}$$

So we have that $\mathscr{L}\{1\} = \dfrac{1}{s}$ for $s > 0$.

Ex: Determine the Laplace transform of $f(t) = e^{at}$, for constant a.

$$\mathscr{L}\left\{e^{at}\right\} = \int_0^\infty e^{-st} e^{at}\, dt = \lim_{N\to\infty} \int_0^N e^{-(s-a)t}\, dt = \lim_{N\to\infty} \left(\frac{-e^{-(s-a)t}}{s-a}\right)\Bigg|_0^N = \frac{1}{s-a}, \text{ for } s > a.$$

Observe that we can consider $f(t) = 1 = e^{0t}$ and the Laplace transforms match up.

Ex: Determine the Laplace transform of $f(t) = \sin(bt)$ $(b \neq 0)$.

$$\mathscr{L}\{\sin(bt)\} = \int_0^\infty e^{-st} \sin(bt)\, dt$$

We will rewrite this as a limit and then perform the classic 'two integration by parts and then divide' method to obtain the following:

$$\mathscr{L}\{\sin(bt)\} = \lim_{N\to\infty} \left(\frac{-se^{-st}\sin(bt) - be^{-st}\cos(bt)}{s^2+b^2}\right)\Bigg|_0^N = \frac{b}{s^2+b^2}$$

the last limit being computed with squeeze theorem.

We now observe an interesting fact about the Laplace transform - that is smooths out jump discontinuities.

Ex: Determine the Laplace transform of

$$f(t) = \begin{cases} 2 & ,0 < t < 5 \\ 0 & ,5 < t < 10 \\ e^t & ,10 < t \end{cases}$$

We need only apply the definition and use properties of integrals that we already know.

$$\mathscr{L}\{f\} = \int_0^5 e^{-st}(2)dt + \int_5^{10} e^{-st}(0)dt + \lim_{N\to\infty} \int_1 0^N e^{-st} e^t\, dt$$

$$= \left(\frac{2}{s} - \frac{2e^{-5s}}{s}\right) + \lim_{N\to\infty} \left(\frac{e^{-(s-1)t}}{s-1}\right)\Bigg|_{10}^N = \frac{2}{s} - \frac{2e^{-5s}}{s} + \frac{e^{-10(s-1)}}{s-1}, \text{ for } s > 1$$

We now describe some important facts about the Laplace transform. Firstly, it is linear. That is, if f_1, f_2 are functions whose Laplace transforms exist, and c is any constant, then

$$\mathscr{L}\{f_1 + f_2\} = \mathscr{L}\{f_1\} + \mathscr{L}\{f_2\} \qquad \mathscr{L}\{cf_1\} = c\mathscr{L}\{f_1\}$$

Both of these come from the fact that the integral is linear.

Ex: Determine $\mathscr{L}\left\{13 + 4e^{6t} - 3\sin(4t)\right\}$.

We begin by using linearity and then using the transforms we already know:

$$\begin{aligned}
\mathscr{L}\left\{13 + 4e^{6t} - 3\sin(4t)\right\} &= \mathscr{L}\{13\} + \mathscr{L}\left\{4e^{6t}\right\} + \mathscr{L}\{-3\sin(4t)\} \\
&= 13\mathscr{L}\{1\} + 4\mathscr{L}\left\{e^{6t}\right\} - 3\mathscr{L}\{\sin(4t)\} \\
&= \frac{13}{s} + \frac{4}{s-6} - \frac{12}{s^2+16}, \text{ for } s > 6
\end{aligned}$$

Suggested HW: P.360 # 1-20, # 31

6.2 Additional Properties

We are now going to examine some general properties of the Laplace transform. The first will be the effect of exponentials on a function.

If the Laplace transform $\mathscr{L}\{f\}(s) = F(s)$ exists for $s > \alpha$, then

$$\mathscr{L}\left\{e^{at}f(t)\right\}(s) = F(s-a) \qquad \text{for } s > \alpha + a$$

This is easy to see by direct computation:

$$\mathscr{L}\left\{e^{at}f(t)\right\} = \int_0^\infty e^{-st}e^{at}f(t)dt = \int_0^\infty e^{-(s-a)t}f(t)\,dt = F(s-a)$$

Ex: Determine $\mathscr{L}\{e^{at}\sin(bt)\}$.

We simply take our known Laplace transform for $\sin(bt)$ and shift it by a:

$$\mathscr{L}\{\sin(bt)\} = \frac{b}{s^2+b^2} \Longrightarrow \mathscr{L}\left\{e^{at}\sin(bt)\right\} = \frac{b^2}{(s-a)^2+b^2}$$

The next thing we will look at will be the effect of the Laplace transform on derivatives of functions. Let f, f' be continuous and of exponential order and both have Laplace transforms $\mathscr{L}\{f\}, \mathscr{L}\{f'\}$, respectively. Then we have that

$$\mathscr{L}\{f'\} = s\mathscr{L}\{f\} - f(0)$$

We can see this by examining the definition of the transform of f' and then performing an integration by parts:

$$\mathscr{L}\{f'\} = \int_0^\infty e^{-st}f'(t)\,dt = \lim_{N\to\infty}\left(e^{-st}f(t)\right)\Big|_0^N + s\int_0^\infty e^{-st}f(t)\,dt = -f(0) + s\mathscr{L}\{f\}$$

We can use this result to obtain an identity for second order derivatives:

$$\mathscr{L}\{f''\} = s\mathscr{L}\{f'\} - f'(0) = s(s\mathscr{L}\{f\} - f(0)) - f'(0) \Longrightarrow \mathscr{L}\{f''\} = s^2\mathscr{L}\{f\} - sf(0) - f'(0)$$

We can then use induction to get the following general expression. Let $f, f', \dots, f^{(n)}$ be continuous of exponential order and all their transforms exist. Then

$$\mathscr{L}\left\{f^{(n)}\right\} = s^n \mathscr{L}\{f\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$$

Ex: Determine $\mathscr{L}\{\cos(bt)\}$ using the derivative rule.
Observe that for $f(t) = \cos(bt)$, $f'(t) = -b\sin(bt)$. So we can apply the derivative rule and linearity of the transform to solve for the transform of $\cos(bt)$:

$$\mathscr{L}\{-b\sin(bt)\} = s\mathscr{L}\{\cos(bt)\} - f(0)$$

We know that $\mathscr{L}\{-b\sin(bt)\} = -b\mathscr{L}\{\sin(bt)\} = \dfrac{-b^2}{s^2+b^2}$. Given that $f(0) = 1$, we can solve for $\mathscr{L}\{\cos(bt)\}$:

$$\frac{-b^2}{s^2+b^2} = s\mathscr{L}\{\cos(bt)\} - 1 \Longrightarrow \mathscr{L}\{\cos(bt)\} = \frac{1}{s}\left(1 - \frac{b^2}{s^2+b^2}\right) = \frac{s}{s^2+b^2}$$

The next property we would like to examine is a sort-of reverse to what we just discovered. That is, since it seem that transforms of derivatives correspond to multiplication by s, we wonder if there is any relation between multiplication by t and derivatives of transforms. In fact, there is such a relationship. Observe the following:

$$\frac{d}{ds}F(s) = \frac{d}{ds}\int_0^\infty e^{-st}f(t)\,dt = \int_0^\infty \frac{d}{ds}(e^{-st})f(t)\,dt$$

This last step of swapping the integral and the derivative should give some cause for concern, but in our case it turns out that things work out fine. From this we can conclude that

$$\frac{dF}{ds} = -\mathscr{L}\{tf(t)\}$$

We can perform the same again, and by induction, we obtain the following relationship:

$$\frac{dF}{ds} = (-1)^n\mathscr{L}\{t^n f(t)\}$$

or perhaps more useful

$$\mathscr{L}\{t^n f(t)\} = (-1)^n\frac{d^nF}{ds^n}$$

Ex: Determine $\mathscr{L}\{t\sin(bt)\}$.
We will take our transform for $\sin(bt)$ and then apply our derivative rule:

$$\mathscr{L}\{t\sin(bt)\} = (-1)\frac{d}{ds}\left(\frac{b}{s^2+b^2}\right) = \frac{2bs}{(s^2+b^2)^2}$$

Ex: Determine $\mathscr{L}\{t\}$.
We will compute this directly using the definition and integration by parts. We leave this to the reader to perform the operations. The result is as follows

$$\mathscr{L}\{t\} = \frac{1}{s^2}$$

We could have done this by viewing $\mathscr{L}\{t\} = \mathscr{L}\{t\cdot(1)\}$. Since we know the transform for 1, we simply use the previous result about derivatives. We can then compute $\mathscr{L}\left\{t^2\right\}$ in a following manner and get

$$\mathscr{L}\left\{t^2\right\} = \frac{2}{s^3}$$

We can perform induction to get the following general result

$$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

Ex: Determine $\mathscr{L}\{t^n e^{at}\}$.
We could compute this a number of ways. We could take the transform of e^{at} and then perform n derivatives - not ideal. Or we could instead take the transform of t^n and use the fact that exponentials shift the transform - a much nicer approach.

$$\mathscr{L}\left\{t^n e^{at}\right\} = \frac{n!}{(s-a)^{n+1}}$$

We can now perform most of the basic transforms of functions. We still need to ask, why were we doing this? We begin to answer this in the next section.

Suggested HW: P.365 # 1-12, # 16-17, # 21, # 24-25

6.3 Inverse Laplace Transforms

Let's look at another way to compute the Laplace transform of $\sin(bt)$. Observe that $y = \sin(bt)$ is a solution to the differential equation $\ddot{y} + b^2 y = 0$. Let's take the Laplace transform of each side of this equation:

$$\mathscr{L}\{\ddot{y}\} + b^2\mathscr{L}\{y\} = 0 \Longrightarrow s^2\mathscr{L}\{y\} - sy(0) - \dot{y}(0) + b^2\mathscr{L}\{y\} = 0$$

Notice that since $\sin(bt)$ is a solution, we have that $y(0) = 0$ and $\dot{y}(0) = b$. Replacing these into our equation we can then solve for $\mathscr{L}\{y\}$.

$$\mathscr{L}\{y\}(s^2 + b^2) - b = 0 \Longrightarrow \mathscr{L}\{y\} = \frac{b}{s^2 + b^2}$$

In this case, we knew the function, but we didn't know the transform. We used the equation to solve for the transform. What would happen if we didn't know the function, but we could still solve for the transform. We would like a way of 'going back' and determining what the original function had to be. To motivate this, consider the following equation.

$$y'' - y = -t \qquad y(0) = 0, y'(0) = 1$$

Let us take the Laplace transform of this equation.

$$\mathscr{L}\{y''\} - \mathscr{L}\{y\} = \mathscr{L}\{-t\} = -\frac{1}{s^2}$$

We can replace the derivative expression and then solve for $\mathscr{L}\{y\}$.

$$\begin{aligned}
-\frac{1}{s^2} &= s^2\mathscr{L}\{y\} - sy(0) - y'(0) - \mathscr{L}\{y\} \\
1 - \frac{1}{s^2} &= \mathscr{L}\{y\}(s^2 - 1) \\
\frac{s^2 - 1}{s^2} &= \mathscr{L}\{y\}(s^2 - 1) \\
\frac{1}{s^2} &= \mathscr{L}\{y\}
\end{aligned}$$

From this we see that $\mathscr{L}\{y\} = \frac{1}{s^2}$. We recall that $\mathscr{L}\{t\} = \frac{1}{s^2}$, so we would like to say that $y = t$. Indeed this solution does work, however, it could be the case that our function was $\begin{matrix} t & , 4 < t \\ 0 & , 0 < t < 4 \end{matrix}$ which also has a Laplace transform of $\frac{1}{s^2}$. This leads us to wanting to make sure we are choosing functions that are continuous.

> Definition: Given $F(s)$, if there exists a function $f(t)$ that is continuous on $[0,\infty)$ such that $\mathscr{L}\{f\} = F$, then we say that $f(t)$ is the inverse Laplace transform of $F(s)$ and we write $f = \mathscr{L}^{-1}\{F\}$.

Ex: Determine $\mathscr{L}^{-1}\left\{\frac{2}{s^3}\right\}$.

$$\frac{2}{s^3} = \frac{2!}{s^{(2)+1}} \Longrightarrow \mathscr{L}^{-1}\left\{\frac{2}{s^3}\right\} = t^2$$

Ex: Determine $\mathscr{L}^{-1}\left\{\frac{3}{s^2+9}\right\}$.

$$\frac{3}{s^2+9} = \frac{3}{s^2+3^2} \Longrightarrow \mathscr{L}^{-1}\left\{\frac{3}{s^2+9}\right\} = \sin(3t)$$

Ex: Determine $\mathscr{L}^{-1}\left\{\frac{s-1}{s^2-2s+5}\right\}$.

It is not immediately obvious which function this arises from. To help, we will complete the square in the denominator.

$$\frac{s-1}{s^2-2s+5} = \frac{s-1}{(s-1)^2+4} \Longrightarrow \mathscr{L}^{-1}\left\{\frac{s-1}{s^2-2s+5}\right\} = e^t\cos 2t$$

An important fact about the inverse Laplace transform is that it is linear

$$\mathscr{L}^{-1}\{F_1+F_2\} = \mathscr{L}^{-1}\{F_1\} + \mathscr{L}^{-1}\{F_2\} \qquad \mathscr{L}^{-1}\{cF\} = c\mathscr{L}^{-1}\{F\}$$

Ex: Determine $\mathscr{L}^{-1}\left\{\frac{5}{s-6} - \frac{6s}{s^2+9} + \frac{3}{2s^2+8s+10}\right\}$.

$$\mathscr{L}^{-1}\left\{\frac{5}{s-6} - \frac{6s}{s^2+9} + \frac{3}{2s^2+8s+10}\right\} = 5\mathscr{L}^{-1}\left\{\frac{1}{s-6}\right\} - 6\mathscr{L}^{-1}\left\{\frac{s}{s^2+9}\right\} + 3\mathscr{L}^{-1}\left\{\frac{1}{2(s+2)^2+2}\right\}$$

$$= 5e^{6t} - 6\cos(3t) + \frac{3}{2}e^{-2t}\sin(t)$$

Often times however, we do not have expressions that already fit into a particular template. We may have to split rational expressions using partial fractions decomposition.

Ex: Determine $\mathscr{L}^{-1}\left\{\frac{7s-1}{(s+1)(s+2)(s-3)}\right\}$.

We begin with partial fractions:

$$\frac{7s-1}{(s+1)(s+2)(s-3)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s-3} \Longrightarrow 7s-1 = A(s+2)(s-3) + B(s+1)(s-3) + C(s+1)(s+2)$$

Picking convenient values for s (i.e. $s=-1,-2,3$), we find that $A=2$, $B=-3$, and $C=1$. Now using the rewritten sum of fractions we can take the inverse Laplace transform of the following expression

$$\mathscr{L}^{-1}\left\{\frac{2}{s+1} - \frac{3}{s+2} + \frac{1}{s-3}\right\} = 2e^{-t} - 3e^{-2t} + e^{3t}$$

Ex: Determine $\mathscr{L}^{-1}\left\{\frac{s^2+9s+2}{(s-1)^2(s+3)}\right\}$.

Again, we proceed by partial fractions:

$$\frac{s^2+9s+2}{(s-1)^2(s+3)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+3} \Longrightarrow s^2+9s+2 = A(s-1)(s+3) + B(s+3) + C(s-1)^2$$

Picking convenient values for s (i.e. $1, -3$ and then 0), we find that $A = 2$, $B = 3$, and $C = -1$. Now using the rewritten sum we have

$$\mathscr{L}^{-1}\left\{\frac{2}{s-1} + \frac{3}{(s-1)^2} - \frac{1}{s+3}\right\} = 2e^t + 3te^t - e^{-3t}$$

Ex: Determine $\mathscr{L}^{-1}\left\{\frac{2s^2+10s}{(s^2-2s+5)(s+1)}\right\}$.

We use partial fractions while observing that we have an irreducible quadratic in the denominator:

$$\frac{2s^2+10s}{(s^2-2s+5)(s+1)} = \frac{As+B}{s^2-2s+5} + \frac{C}{s+1} \Longrightarrow 2s^2+10s = (As+B)(s+1) + C(s^2-2s+5)$$

Picking convenient values $s = -1, 0$ we see that $B = 5$ and $C = -1$. Equating the coefficients for the s^2 term we see that $A = 3$. Now we have the following

$$\mathscr{L}^{-1}\left\{\frac{3s+5}{s^2-2s+5} - \frac{1}{s+1}\right\} = \mathscr{L}^{-1}\left\{\frac{3s+5}{(s-1)^2+4} - \frac{1}{s+1}\right\} = \mathscr{L}^{-1}\left\{\frac{3(s-1)+8}{(s-1)^2+4} - \frac{1}{s+1}\right\}$$

We can split the first fraction now and we have

$$\mathscr{L}^{-1}\left\{\frac{3(s-1)}{(s-1)^2+4} + \frac{8}{(s-1)^2+4} - \frac{1}{s+1}\right\} = 3e^t\cos(2t) + 4e^t\sin(2t) - e^{-t}$$

Suggested HW: P.374 # 1-30, # 31-32, # 33-36

6.4 Solving IVPs

We now illustrate the main purpose of building all the tools we have for Laplace transforms through solving IVP's.

Ex: Solve $y'' - 2y' + 5y = -8e^{-t} \qquad y(0) = 2,\ y'(0) = 12.$
We begin by taking the Laplace transform of both sides

$$\mathscr{L}\{y''\} - 2\mathscr{L}\{y'\} + 5\mathscr{L}\{y\} = -\frac{8}{s+1} \Longrightarrow [s^2\mathscr{L}\{y\} - sy(0) - y'(0)] - 2[s\mathscr{L}\{y\} - y(0)] + 5\mathscr{L}\{y\} = -\frac{8}{s+1}$$

Next we isolate $\mathscr{L}\{y\}$:

$$\mathscr{L}\{y\}(s^2-2s+5) = 2s+8-\frac{8}{s+1} = \frac{2s^2+10s}{s+1} \Longrightarrow \mathscr{L}\{y\} = \frac{2s^2+10s}{(s^2-2s+5)(s+1)}$$

Now we need only determine the inverse Laplace transform for the right hand side. By a complete coincidence (ahem), we already did this in the last example above. Thus, the solution to the IVP is
$y(t) = 3e^t\cos(2t) + 4e^t\sin(2t) - e^{-t}$.

It might be the case, as we have seen, that the initial data is not set at $x = 0$, which is required for using Laplace transforms. The way to handle situations like this is to perform a substitution which shifts the initial data to 0. For example

$$z''(t) - 2z'(t) + 5z(t) = -8e^{\pi - t} \qquad z(\pi) = 2,\ z'(\pi) = 12.$$

To handle this we first perform the transformation $\tau = t - \pi$. This way, when $\tau = 0$, $t = \pi$. And so we have $y(\tau) = z(t - \pi)$ (or $y(\tau + \pi) = z(t)$) which makes our equation

$$y''(\tau) - 2y'(\tau) + 5y(\tau) = -8e^{\pi - (\tau + \pi)} = -8e^{-\tau} \qquad y(0) = 2,\ y'(0) = 12.$$

This can now be solved with Laplace transforms, which we have already done. We would only need to replace in our solution every τ with $t - \pi$.

Ex: Solve $y'' + 6y' + 5y = 12e^t \qquad y(0) = -1\ y'(0) = 7.$
We begin again by taking the Laplace transform

$$\mathscr{L}\{y''\} + 6\mathscr{L}\{y'\} + 5\mathscr{L}\{y\} = \frac{12}{s-1} \Longrightarrow [s^2\mathscr{L}\{y\} - sy(0) - y'(0)] + 6[s\mathscr{L}\{y\} - y(0)] + 5\mathscr{L}\{y\} = \frac{12}{s-1}$$

Now we use our initial data and solve for $\mathscr{L}\{y\}$

$$\mathscr{L}\{y\}(s^2+6s+5) = 1 - s + \frac{12}{s-1} = \frac{-s^2+2s+11}{(s-1)} \Longrightarrow \mathscr{L}\{y\} = \frac{-s^2+2s+11}{(s^2+6s+5)(s-1)} = \frac{-s^2+2s+11}{(s+1)(s+5)(s-1)}$$

We perform partial fractions on the right hand side to get

$$\mathscr{L}\{y\} = -\frac{1}{s+1} - \frac{1}{s+5} + \frac{1}{s-1}$$

Finally, we compute the inverse Laplace transform for each and have

$$y(t) = -e^{-t} - e^{-5t} + e^t$$

Ex: Solve $y'' - 7y' + 10y = 9\cos(t) + 7\sin(t) \qquad y(0) = 5,\ y'(0) = -4.$

$$\mathscr{L}\{y''\} - 7\mathscr{L}\{y'\} + 10\mathscr{L}\{y\} = \frac{9s+7}{s^2+1} \Longrightarrow [s^2\mathscr{L}\{y\} - sy(0) - y'(0)] - 7[s\mathscr{L}\{y\} - y(0)] + 10\mathscr{L}\{y\} = \frac{9s+7}{s^2+1}$$

$$\Longrightarrow \mathscr{L}\{y\}(s^2 - 7s + 10) = \frac{(5s-39)(s^2+1) + 9s + 7}{s^2+1} \Longrightarrow \mathscr{L}\{y\} = \frac{(5s-39)(s^2+1) + 9s + 7}{(s-5)(s-2)(s^2+1)}$$

We perform partial fractions to get

$$\mathscr{L}\{y\} = \frac{-4}{s-5} + \frac{8}{s-2} + \frac{s}{s^2+1}$$

Now we find the inverse Laplace transform

$$y(t) = -4e^{5t} + 8e^{2t} + \cos(t)$$

There is a useful fact about Laplace transforms that we haven't needed o use just yet, but it is the following:

$$\text{Given } \mathscr{L}\{y\}(s) = F(s), \quad \lim_{s\to\infty} F(s) = 0$$

Take a look back at all the transforms we have computed and you'll notice this fact occurring in each of them. This gives us a way to sort-of check if we have the right transform.

Ex: Solve $y'' + 2ty' - 4y = 1 \qquad y(0) = 0,\, y'(0) = 0.$

Notice that this is the first instance that we've encountered a differential equation with non-constant coefficients that isn't first order. With first order equations we had to solve them with integration factors. With this, we will solve it using Laplace transforms.

$$\mathscr{L}\{y''\} + 2\mathscr{L}\{ty'\} - 4\mathscr{L}\{y\} = \frac{1}{s} \Longrightarrow [s^2\mathscr{L}\{y\} - sy(0) - y'(0)] - 2\frac{d}{ds}\mathscr{L}\{y'\} - 4\mathscr{L}\{y\} = \frac{1}{s}$$

We will use our initial data and replace $\mathscr{L}\{y\} = F(s)$.

$$s^2F - 2\frac{d}{ds}(sF) - 4F = \frac{1}{s} \Longrightarrow -2sF' + (s^2 - 6)F = \frac{1}{s} \Longrightarrow F' + \left(-\frac{s}{2} + \frac{3}{s}\right)F = -\frac{1}{2s^2}$$

This is a first order equation which can be solved via integration factor. The integration factor is $\mu = s^3e^{-s^2}$. Our equation then becomes

$$s^3e^{-s^2}F = \int -\frac{1}{2}se^{-s^2}\,ds = \frac{1}{4}e^{-s^2} + C \Longrightarrow F = \frac{1}{4s^3} + \frac{Ce^{s^2}}{s^3}$$

Now we use the fact that the Laplace transform approaches 0 as $s \to \infty$ to see that C must be 0. Hence, we have that $\mathscr{L}\{y\} = \frac{1}{4s^3}$ so that $y(t) = \frac{1}{8}t^2$.

Suggested HW: P.382 # 1-22, # 25-28, # 29, # 31

6.5 Transforms of Discontinuities

We are leading up to convolution, but before we do, we will look at what happens when we take Laplace transforms of functions with discontinuities.

Definition: The unit step function $u(t)$ is defined by

$$u(t) := \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

More commonly, by shifting we get

$$u(t-a) := \begin{cases} 0, & t < a \\ 1, & t > a \end{cases}$$

Also, we can change the height by multiplying by a constant

$$Mu(t-a) := \begin{cases} 0, & t < a \\ M, & t > a \end{cases}$$

The step function serves as a sort-of switch, when multiplied to another function. It has the ability to turn on and off a function at a particular time a.

If we want to describe a particular section of a function, we will want to use what is sometimes called a square function

$$Sq_{a,b}(t) := u(t-a) - u(t-b) = \begin{cases} 0, & t < a \\ 1, & a < t < b \\ 0, & t > b \end{cases}$$

This function acts as a on and off switch for functions. It 'turns on' the function at time a and 'turns off' the function at time b.

Let's see what happens when we take the Laplace transform of these functions. Take $a \geq 0$.

$$\mathscr{L}\{u(t-a)\} = \int_0^\infty e^{-st}u(t-a)\,dt = \int_a^\infty e^{-st}\,dt = \frac{e^{-as}}{s}$$

$$\mathscr{L}\{Sq_{a,b}\} = \mathscr{L}\{u(t-a) - u(t-b)\} = \frac{e^{-as} - e^{-bs}}{s}$$

We can now use these 'switches' to see what happens with Laplace transforms of these piecewise continuous functions. Let $F(s) = \mathscr{L}\{f\}$, then

$$\mathscr{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$$

and for inverse Laplace transforms,

$$\mathscr{L}^{-1}\left\{e^{-as}F(s)\right\} = f(t-a)u(t-a)$$

Now often times, we do not have a function that is also shifted by a. We would then need to rewrite the function to be able to use this fact. That is if we want to compute $\mathscr{L}\{g(t)u(t-a)\}$, we identify $g(t) = f(t-a)$ so that we can use our previous result.

$$\mathscr{L}\{g(t)u(t-a)\} = \mathscr{L}\{f(t-a)u(t-a)\} = e^{-as}\mathscr{L}\{f(t)\} = e^{-as}\mathscr{L}\{g(t+a)\}$$

Ex: Compute $\mathscr{L}\left\{t^2u(t-1)\right\}$

$$\mathscr{L}\left\{t^2u(t-1)\right\} = e^{-s}\mathscr{L}\left\{(t+1)^2\right\} = e^{-s}\mathscr{L}\left\{t^2 + 2t + 1\right\} = e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right)$$

Ex: Compute $\mathscr{L}\{\cos(t)u(t-pi)\}$

$$\mathscr{L}\{\cos(t)u(t-\pi)\} = e^{-\pi s}\mathscr{L}\{\cos(t+\pi)\} = e^{-\pi s}\mathscr{L}\{-\cos(t)\} = -e^{-\pi s}\left(\frac{s}{s^2+1}\right)$$

We now look at other functions which potentially have discontinuities.

Definition: A function $f(t)$ is called periodic with period T if $f(t+T) = f(t)$ for all t.

Since the function repeats itself every period, it is reasonable to assume that the Laplace transform over a single period should be related to the overall transform. Let $F_T(s)$ be the transform over a single period, i.e. $F_T(s) = \int_0^T e^{-st}f(t)\,dt$. Observe the following

$$F(s) = \int_0^\infty e^{-st}f(t)\,dt = \int_0^T e^{-st}f(t)\,dt + \int_T^\infty e^{-st}f(t)\,dt = F_T(s) + \mathscr{L}\{f(t)u(t-T)\} = F_T(s) + e^{-Ts}\mathscr{L}\{f(t+T)\}$$

Since f is periodic, we have then that

$$F(s) = F_t(s) + e^{-Ts}\mathscr{L}\{f(t)\}\,F_T(s) + e^{-Ts}F(s)$$

This means that the relation between the full transform and the period transform is

$$F_T(s) = F(s)(1 - e^{-Ts}) \quad \text{or} \quad F(s) = \frac{F_T(s)}{1 - e^{-Ts}}$$

Our final fact will be one involving noninteger powers of t. What should we have if we want $\mathscr{L}\{t^p\}$ for $p > 0$ not necessarily an integer? We'd like some notion of factorial that works for nonintegers. Indeed there is a generalized notion called the Gamma function:

Definition:

$$\Gamma(t) := \int_0^\infty e^{-u} u^{t-1}\, du, \quad t > 0$$

Some facts about this function

- $\Gamma(t+1) = t\Gamma(t)$
- $\Gamma(n+1) = n!$ for n an integer

With this function in mind, we can define the logical extension of the Laplace transform for the function $f(t) = t^p$.

$$\mathscr{L}\{t^p\} = \frac{\Gamma(p+1)}{s^{p+1}}$$

Suggested HW: P.393 # 5-6, # 11-18, # 29, # 31

6.6 Convolution

Suppose we are solving a differential equation with a forcing function. We get down to, say $\mathscr{L}\{y\} = F(s)G(s)$. We cannot say that the desired function is just the product of the inverse transforms of $F(s)$ and $G(s)$. This is mainly because a product of integrals is not the integral of a product. We can however define a different multiplication type of operation called the convolution of two functions.

Definition: Let $f(t)$ and $g(t)$ be piecewise continuous on $[0, \infty)$. The convolution of $f(t)$ and $g(t)$, denoted $f * g$ is defined by

$$(f * g)(t) = \int_0^t f(t-v)g(v)\, dv.$$

Ex: Compute $t * t^2$.

$$t * t^2 = \int_0^t (t-v)v^2\, dv = \int_0^t (tv^2 - v^3) dv = \frac{t^4}{3} - \frac{t^4}{4} = \frac{t^4}{12}$$

Notice that $1 * 1 = t \neq 1$, and also $1 * f \neq f$, in general. So convolution isn't quite a multiplication, but it shares similar properties:

- $f * g = g * f$
- $f * (g + h) = (f * g) + (f * h)$
- $(f * g) * h = f * (g * h)$
- $f * 0 = 0$

With this in mind, we can now address the question of what the Laplace transform of a convolution is. The answer is particularly convenient.

Let $f(t)$ and $g(t)$ be piecewise continuous on $[0,\infty)$ with $F(s)=\mathscr{L}\{f\}$ and $G(s)=\mathscr{L}\{g\}$. Then

$$\mathscr{L}\{f*g\}=F(s)G(s) \qquad \text{and conversely} \qquad \mathscr{L}^{-1}\{F(s)G(s)\}=(f*g)(t)$$

We can now use this tool to solve more delicate IVP's

$$y''+y=g(t) \qquad y(0)=1,\ y'(0)=0$$

We take the Laplace transform of both sides:

$$s^2F(s)-s+F(s)=G(s)\Longrightarrow F(s)=\frac{G(s)+s}{s^2+1}=\frac{1}{s^2+1}G(s)+\frac{s}{s^2+1}$$

Now we take the inverse Laplace transform and we get

$$y(t)=\mathscr{L}^{-1}\left\{\frac{s}{s^2+1}\right\}+\mathscr{L}^{-1}\left\{\frac{1}{s^2+1}G(s)\right\}=\cos(t)+\sin(t)*g(t)=\cos(t)+\int_0^t\sin(t-v)g(v)\,dv$$

Ex: Compute $\mathscr{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\}$.

Notice that $\frac{1}{(s^2+1)^2}=\left(\frac{1}{s^2+1}\right)\left(\frac{1}{s^2+1}\right)$. So we must have that

$$\mathscr{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\}=\sin(t)*\sin(t)=\int_0^t\sin(t-v)\sin(v)dv$$

We can use a difference identity and then integrate to get

$$=-\frac{1}{4}\sin(t)\left(\cos(2v)\right)\big|_0^t-\frac{1}{2}\cos(t)\left(v-\frac{1}{2}\sin(2v)\right)\bigg|_0^t=\frac{1}{2}(\sin(t)-t\cos(t))$$

Ex: Compute $\mathscr{L}^{-1}\left\{\frac{1}{(s-a)(s-b)}\right\}$.

Notice that we could compute partial fractions, but instead we will view this as a convolution

$$\frac{1}{(s-a)(s-b)}=\left(\frac{1}{s-a}\right)\left(\frac{1}{s-b}\right)\Longrightarrow\mathscr{L}^{-1}\left\{\left(\frac{1}{s-a}\right)\left(\frac{1}{s-b}\right)\right\}=e^{at}*e^{bt}$$

So we need only compute this convolution

$$e^{at}*e^{bt}=\int_0^t e^{a(t-v)}e^{bv}\,dv=\int_0^t e^{at}e^{(b-a)v}\,dv$$

$$=e^{at}\left(\frac{1}{b-a}e^{(b-a)v}\right)\bigg|_0^t=e^{at}\left(\frac{1}{b-a}e^{(b-a)t}-\frac{1}{b-a}\right)=\frac{e^{bt}-e^{at}}{b-a}$$

Suggested HW: P.403 # 1-12, # 13, # 14, # 31

6.7 Dirac Delta

Definition: The Dirac delta function $\delta(t)$ is defined with the following properties

- $\delta(t) = \begin{cases} 0, & t \neq 0 \\ `\infty', & t = 0 \end{cases}$
- For any continuous function $f(t)$ on an open interval containing 0
$$\int_0^\infty f(t)\delta(t)\,dt = f(0)$$
- By shifting the input, $\delta(t-a) = 0$ for $t \neq a$ so that
$$\int_0^\infty f(t)\delta(t-a)\,dt = f(a)$$

The first thing to note is that $\delta(t-a)$ is not a function, just by its definition. It is considered a generalized function, or a function in the sense of distributions (something we don't discuss in this class). However, it is useful when describing the effects of impulse on a system.

Impulse is defined to be the following

$$\text{Impulse } = \int_{t_0}^{t_1} F(t)\,dt$$

where the force $F(t)$ is applied from time t_0 to time t_1. Using Newton's laws, we can describe the impulse as the change in momentum over this time period. If the same impulse is applied but over a shorter time interval, it necessitates that the average force applied must get greater and greater i.e.

$$\int_{t_0}^{t_n} F_n(t)\,dt = \int_{t_0}^{t_1} F(t)\,dt$$

If we allow the time interval to approach 0, we see that the force applied $F_n(t)$ must tend toward ∞. We can normalize the expression to have size 1 so that

$$\int_{-\infty}^{\infty} F_(t)\,dt = 1 \qquad \text{for all } n$$

Thus, we see that this description leads us to say that the limiting 'function' of the F_n's is $\delta(t)$.

An interesting relationship between the delta function and the step function can be seen using the Laplace transform.

$$\mathscr{L}\{\delta(t-a)\} = \int_0^\infty e^{-st}\delta(t-a)\,dt = e^{-as}, \text{ for } \; a > 0$$

Notice that

$$\int_{-\infty}^{t} \delta(x-a)\,dx = \begin{cases} 0, & t < a \\ 1, & t > a \end{cases} = u(t-a)$$

So if we were to 'differentiate' both sides, with respect to t, we would find that

$$\delta(t-a) = u'(t-a)$$

Ex: Solve $y'' + 9y = 3\delta(t-\pi) \quad y(0) = 1, y'(0) = 0$.
We begin by taking the Laplace transform and isolating $F(s)$

$$s^2F(s) - 2(1) - (0) + 9F(s) = 3e^{-\pi s} \Longrightarrow F(s)(s^2+9) = 3e^{-\pi s} + s \Longrightarrow F(s) = e^{-\pi s}\frac{3}{s^2+9} + \frac{s}{s^2+9}$$

We now take the inverse Laplace transform whilst remembering that doing so brings in the step function

$$y(t) = \sin(3(t-\pi))u(t-\pi) + \cos(3t)$$

Suggested HW: P.410 # 1-12, # 13-20, # 21-24 (ignore graph sketching)

7 Systems of Equations

We will now take a look at solving systems of differential equations. We begin by using the Laplace transform and then we will move on to more indirect ways of solving systems.

7.1 Laplace Systems

A system of differential equations is a set of equations with an equal number of unknown solutions, each depending on the same independent variable. For example,

$$\begin{cases} x'(t) = 4x + 5y, & x(0) = 2 \\ y'(t) = 2x - 3y, & y(0) = 1 \end{cases}$$

Notice that this has two differential equations in terms of x, y where both of them depend on the independent variable t. It also comes equipped with a pair of initial data, enough to solve for necessary constants in each solution. We will solve this system by taking Laplace transforms of both equations and then performing the usual algebraic manipulations required to solve a classic pre-calculus system of equations.

Ex:

$$\begin{cases} x' - 2y = 4t, & x(0) = 4 \\ y' + 2y - 4x = -4t - 2, & y(0) = -5 \end{cases}$$

We begin by taking the Laplace transform of both equations. We will denote $\mathscr{L}\{x\} = X$ and $\mathscr{L}\{y\} = Y$.

$$\begin{cases} sX - 4 - 2Y &= \dfrac{4}{s^2} \\ sY + 5 + 2Y - 4X &= -\dfrac{4}{s^2} - \dfrac{2}{s} \end{cases} \Longrightarrow \begin{cases} sX - 2Y &= \dfrac{4+4s^2}{s^2} \\ (s+2)Y - 4X &= -\dfrac{5s^2+2s+4}{s^2} \end{cases}$$

This is simply an algebraic system in two variables, and as such can be solved by pre-calculus methods. We will eliminate the Y first, say, and then solve for X. We begin by multiplying the first equation by $(s+2)$ and the second by 2, and then we will add the two equations.

$$\begin{cases} s(s+2)X - 2(s+2)Y &= \dfrac{(4+4s^2)(s+2)}{s^2} \\ 2(s+2)Y - 8X &= -\dfrac{10s^2+4s+8}{s^2} \end{cases} \Longrightarrow (s^2+2s-8)X = \frac{(4s^3+8s^2+4s+8) - (10s^2+4s+8)}{s^2}$$

$$(s+4)(s-2)X = \frac{4s^3 - 2s^2}{s^2} = 4s - 2 \Longrightarrow X = \frac{4-2s}{(s+4)(s-2)}$$

We now perform partial fractions and then take the inverse Laplace transform.

$$X = \frac{3}{s+4} + \frac{1}{s-2} \Longrightarrow x(t) = 3e^{-4t} + e^{2t}$$

Now we need to solve for $y(t)$. Instead of solving for Y, we will use the original equations and solve for y in terms of x and x'. Using the first of the original system we have that

$$y(t) = \frac{1}{2}(x' - 4t) = \frac{1}{2}(-12e^{-4t} + 2e^{2t} - 4t) = -6e^{-4t} + e^{2t} - 2t$$

This method would work for any finite number of equations and unknown solutions.

Ex:

$$\begin{cases} x' = 3x - 2y, & x(0) = 1 \\ y' = 3y - 2x, & y(0) = 1 \end{cases}$$

Again, we begin by taking the Laplace transform of both equations.

$$\begin{cases} sX - 1 &= 3X - 2Y \\ sY - 1 &= 3Y - 2X \end{cases} \Longrightarrow \begin{cases} (s-3)X + 2Y &= 1 \\ (s-3)Y + 2X &= 1 \end{cases}$$

We will eliminate X, say, this time. We multiply the first equation by -2 and the second by $s-3$ and then add the two equations.

$$\begin{cases} -2(s-3)X - 4Y &= -2 \\ (s-3)^2Y + 2(s-3)X &= s-3 \end{cases} \Longrightarrow (s^2 - 6s + 5)Y = s - 5$$

We solve for Y and then take the inverse Laplace transform

$$Y = \frac{s-5}{(s-5)(s-1)} = \frac{1}{s-1} \Longrightarrow y(t) = e^t$$

Again, to solve for $x(t)$, we will take one of the original equations (the second) and then solve for x by replacing y and y'.

$$x(t) = \frac{1}{2}(3y - y') = \frac{1}{2}(3e^t - e^t) = e^t$$

Which shouldn't be surprising since the original system of equations is completely symmetric in x and y.

Ex:

$$\begin{cases} x' = y + \sin(t), & x(0) = 2 \\ y' = x + 2\cos(t), & y(0) = 0 \end{cases}$$

As before, we take the Laplace transform of both equations.

$$\begin{cases} sX - 2 &= Y + \dfrac{1}{s^2+1} \\ sY &= X + \dfrac{2s}{s^2+1} \end{cases} \Longrightarrow \begin{cases} sX - Y &= 2 + \dfrac{1}{s^2+1} \\ sY - X &= \dfrac{2s}{s^2+1} \end{cases}$$

We will eliminate Y, say, by multiplying the first equation by s and then adding the two equations

$$\begin{cases} s^2X - sY &= 2s + \dfrac{s}{s^2+1} \\ sY - X &= \dfrac{2s}{s^2+1} \end{cases} \Longrightarrow (s^2 - 1)X = \frac{2s(s^2+1) + 3s}{s^2+1} = \frac{2s^3 + 5s}{s^2+1}$$

We solve for X and then perform partial fractions before taking the inverse Laplace transform

$$X = \frac{2s^3 + 5s}{(s-1)(s+1)(s^2+1)} = \frac{7}{4(s-1)} + \frac{7}{4(s+1)} - \frac{3s}{2(s^2+1)} \Longrightarrow x(t) = \frac{7}{4}e^t + \frac{7}{4}e^{-t} - \frac{3}{2}\cos(t)$$

Again, to find y we need only use the first original equation and solve for y

$$y(t) = x' - \sin(t) = \frac{7}{4}e^t - \frac{7}{4}e^{-t} + \frac{1}{2}\sin(t)$$

Suggested HW: P.413 # 1-6, # 9-19

7.2 Systems by Substitution

We now try solving systems by a method different from Laplace. We will rewrite these systems in such a way that we will solve a single differential equation with our old techniques. Consider the following example

$$\begin{cases} x' = 3x - 4y + 1, & x(0) = 1 \\ y' = 4x - 7y + 10t, & y(0) = 1 \end{cases}$$

We will solve this differential equation by a seemingly circuitous route - by taking a further derivative. We will differentiate the first equation so that we can solve for y' entirely in terms of x and x'. That is,

$$x'' = 3x' - 4y' \Longrightarrow 4y' = 3x' - x''$$

we will now use this fact and the first equation to substitute for y and y' in the second equation. Thus, creating a second order equation entirely in terms of x. We begin by multiplying the second equation by 4 and then substituting in

$$3x' - x'' = 16x - 7(3x - x' + 1) + 40t \Longrightarrow x'' + 4x' - 5x = 7 - 40t$$

We can now solve this equation for x by old methods. We begin by finding the homogeneous solution. The roots of the characteristic equation are $r = -5, 1$, so our homogeneous solution is

$$x_h(t) = c_1 e^{-5t} + c_2 e^t.$$

We need a particular solution now - we will use undetermined coefficients. Our guess for the particular solution is $x_p = At + B$. Taking derivatives and subbing into the equation we see that $A = 8$ and $B = 5$. So that our general solution is

$$x(t) = c_1 e^{-5t} + c_2 e^t + 8t + 5.$$

To find $y(t)$, we again go back to the original equations and solve for y

$$4y = 3x - x' + 1 = 3(c_1 e^{-5t} + c_2 e^t + 8t + 5) - (-5c_1 e^{-5t} + c_2 e^t + 8) + 1 \Longleftarrow y(t) = 2c_1 e^{-5t} + c_2 e^t + 6t + 2$$

We now only need to solve for c_1 and c_2. We have our initial conditions which yield the following system

$$\begin{cases} c_1 + c_2 & = & -4 \\ 2c_1 + c_2 & = & -1 \end{cases}$$

Solving this system gives $c_1 = 3$ and $c_2 = -7$. Hence our two solutions are

$$\begin{aligned} x(t) &= 3e^{-5t} - 7e^t + 8t + 5 \\ y(t) &= 6e^{-5t} - 7e^t + 6t + 2 \end{aligned}$$

Let's now compare this method to the method with Laplace transforms. Recall this example

$$\begin{cases} x' = 3x - 2y, & x(0) = 1 \\ y' = 3y - 2x, & y(0) = 1 \end{cases}$$

We will now solve this using the methods of substitution. We will differentiate the first equation and use it to solve for y' in terms of x'' and x'.

$$x'' = 3x' - 2y' \Longrightarrow 2y' = 3x' - x''$$

Now using this and the first equation we will substitute in the second equation (multiply by 2 first) to get a second order equation in terms of x.

$$3x' - x'' = 3(3x - x') - 4x \Longrightarrow x'' - 6x' + 5x = 0$$

This is a homogeneous equation, so we can easily see that the roots of the characteristic equation are $r = 5, 1$. Hence, the general solution is

$$x(t) = c_1 e^{5t} + c_2 e^t.$$

We can use the first equation to solve for $y(t)$

$$2y = 3x - x' = 3(c_1e^{5t} + c_2e^t) - (5c_1e^{5t} + c_2e^t) \Longrightarrow y(t) = -c_1e^{5t} + c_2e^t$$

We now use the initial conditions to solve for c_1 and c_2. The initial conditions give the system

$$\begin{array}{rcl} c_1 + c_2 & = & 1 \\ -c_1 + c_2 & = & = 1 \end{array}$$

This gives $c_1 = 0$ and $c_2 = 1$. Hence, our solutions are

$$\begin{array}{rcl} x(t) & = & e^t \\ y(t) & = & e^t \end{array}$$

This isn't surprising since we already found these solutions, but it's nice to see another method working well. It's up to you to decide which method is more efficient. Most likely it will depend on the situation.

Ex:

$$\begin{cases} x' = y + \sin(t), & x(0) = 2 \\ y' = x + 2\cos(t), & y(0) = 0 \end{cases}$$

Unfortunately, this method of substitution does not work in every circumstance. Consider the example

$$\begin{cases} x'' + y'' - x' & = \quad 2t \\ yx'' + y' - x + y & = \quad -1 \end{cases}$$

Try as much as we might, there isn't a good way to write one differential equation only in terms of a single variable. Mainly because we don't have a way of isolating y'. If we had another equation, we might be able to, or if the first equation also had a y. However, we can still solve this using Laplace methods. This potentially is a point in favor of the Laplace method since it can handle more situations.

It's important to mention that there are other methods of solving systems of differential equations. A common method is to use matrix techniques. However, you will have to wait for a linear algebra course to learn about these.

Suggested HW: P.413 # 1-6, # 9-19 (yes same problems)

7.3 Single Equations by Systems

Our last topic of study is potentially an unusual method for solving single differential equations. We will take a single equation and rewrite it as a system of differential equations.

Consider the differential equation below

$$y'' + 2y' - 8y = 0 \qquad y(0) = 3, \ y'(0) = -12$$

We will solve this equation by defining a substitution and rewriting it as a system of equations. Let $u = y$ and $v = y'$. Then we have

$$\begin{array}{lcl} u = y & \Rightarrow & u' = y' = v \\ v = y' & \Rightarrow & v' = y'' = 8y - 2y' = 8u - 2v \end{array}$$

So we now have the system

$$\begin{cases} u' = v, & u(0) = 3 \\ v' = 8u - 2v, & v(0) = -12 \end{cases}$$

We can now solve this system by either of the two methods we have see so far. However, since this came from a single equation, it probably makes more sense to solve this with Laplace transforms. Keep in mind that we want to solve for y, which in this system is u. So when we eliminate a variable, we should eliminate v. We take the Laplace transform (with $\mathscr{L}\{u\} = U$ and $\mathscr{L}\{v\} = V$) and get

$$\begin{cases} sU - 3 &= V \\ sV + 12 &= 8U - 2V \end{cases} \implies \begin{cases} sU - V &= 3 \\ (s+2)V - 8U &= -12 \end{cases}$$

We will eliminate V, so we will multiply the first equation by $(s+2)$ and then add the equations

$$\begin{cases} s(s+2)U - (s+2)V &= 3(s+2) \\ (s+2)V - 8U &= -12 \end{cases} \implies (s^2 + 2s - 8)U = 3s - 6$$

We now solve for U and then take the inverse Laplace transform

$$U = \frac{3(s-2)}{(s+4)(s-2)} = \frac{3}{s+4} \implies u(t) = y(t) = 3e^{-4t}$$

Now this may seem like a longer method than our earlier techniques, and for some you would be right. However, you can probably see how this can be of more use if there are higher order equations. It is particularly useful in conjunction with matrix methods for solving systems. Alas, again you will have to wait for linear algebra to be able to see this.

Ex: Solve $y'' - 4y' - 4y = 0 \qquad y(0) = 1,\ y'(0) = 1.$

We begin with the substitutions necessary to write this as a system of equations

$$\begin{aligned} u = y \quad &\Rightarrow \quad u' = y' = v \\ v = y' \quad &\Rightarrow \quad v' = y'' = 4y' - 4y = 4v - 4u \end{aligned}$$

So we have the system

$$\begin{cases} u' = v, & u(0) = 1 \\ v' = 4v - 4u, & v(0) = 1 \end{cases}$$

We will take the Laplace transform of both equations

$$\begin{cases} sU - 1 &= V \\ sV - 1 &= 4V - 4U \end{cases} \implies \begin{cases} sU - V &= 1 \\ (s-4)V + 4U &= 1 \end{cases}$$

We will eliminate V, so we will multiply the first equation by $(s-4)$ and then add the equations

$$\begin{cases} s(s-4)U - (s-4)V &= (s-4) \\ (s-4)V + 4U &= 1 \end{cases} \implies (s^2 - 4s + 4)U = s - 3$$

We now solve for U, perform partial fractions, and then take the inverse Laplace transform

$$U = \frac{(s-3)}{(s-2)^2} = \frac{1}{s-2} - \frac{1}{(s-2)^2} \implies u(t) = y(t) = e^{2t} - te^{2t}$$

Ex: Solve $y'' + y = 2e^{-t} \qquad y(0) = 0,\ y'(0) = 0.$

Ex: Solve $y'' + y' - 12y = e^t + e^{2t} - 1 \qquad y(0) = 1,\ y'(0) = 3.$

Suggested HW: P.??? Pick your favourite IVP from any of the past sections.

Probably best to keep to no higher than third order for simplicity sake.

I hope you have enjoyed the course. I hope you have gained an appreciation for the finesse that goes into solving differential equations and that you are inspired to learn more mathematics and see the beautiful things it has to offer. Until next class....

— Jeramiah.