Research Statement

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Current Research

My main mathematical research interests are in the study of operator algebras and functional analysis. Specifically, I study structures originating from Clifford algebras. I have studied enveloping operator algebras arising from Clifford algebras as well as the Clifford Hilbert Space and Clifford antidual. A Clifford algebra $C(V)$ is formed, via the universal mapping property, from an inner product space $(V, (\cdot|\cdot))$ (real or complex) and a map $\epsilon : V \to A$ taking V to a complex unital associative algebra A and satisfying the Clifford relations $\epsilon(v)^2 = (v|v)$ for all $v \in V$. We can define a trace on the Clifford algebra and form an inner product on the Clifford algebra. We then complete $C(V)$ to a Hilbert space, $\mathbb{H}[V]$, via the norm arising from the trace. We construct the left regular representation λ, of $C(V)$ in $B(\mathbb{H})$, the space of bounded operators on $\mathbb{H}$, which is the image of $C(V)$ under λ in $B(\mathbb{H})$, i.e. $\lambda(C(V))$. Depending on what norm we close this space under, we obtain various operator algebras. If we take the closure under the norm given by the trace, which is a C^* norm, we obtain the C^* Clifford algebra. If we instead take the closure in the weak or strong operator topologies, we obtain the von Neumann Clifford algebra. We can instead consider the space of antilinear functionals mapping $C(V) \to \mathbb{C} : a \mapsto \langle \cdot | a \rangle$. This is the Clifford antidual $C'(V)$ and has some interesting algebraic properties.

There is one particular question that I am interested in with regards to each of these; namely the notion of 'twisted duality'. In the general Clifford algebra, there is associated to it a natural grading automorphism γ, which makes $C(V)$ into a superalgebra. For a subspace $W \leq V$, we can consider the supercommutant (or graded commutant)

$$C(W)^\gamma = \{a \in C(V) | (\forall w \in W), wa = \gamma(a)w\}.$$

That is, the elements of the Clifford algebra which graded commute with every element of the subspace. The question of 'twisted duality' is as follows: given a subspace $W \leq V$, we ask if the graded commutant of $C(W)$ is equal to the Clifford algebra generated by the orthogonal space $W^\perp$, i.e. is

$$C(W)^\gamma = C(W^\perp).$$

In the case of the general Clifford algebra, this is true (for both real and complex inner product spaces).

My research is focused on determining if the statement of 'twisted duality' is true in these enveloping operator algebras, the Clifford Hilbert space, and the Clifford antidual. It has already been shown that for real C^* Clifford algebras, 'twisted duality' holds, i.e. $C[W]^\gamma = C[W^\perp]$; I have shown in my research that for real von Neumann Clifford algebras, 'twisted duality' holds, i.e. $\mathcal{A}[W]^\gamma = \mathcal{A}\left[W^\perp\right]$. I have also shown that it holds in the Clifford Hilbert space. That is, for a subspace $W \leq V$ we can take the Clifford algebra generated by the orthogonal space $W^\perp$ and complete it with respect to the tracial inner product on $C(V)$. This gives a subspace of the Hilbert space $\mathbb{H}[V]$ and I have shown that indeed, $\mathbb{H}[W]^\gamma = \mathbb{H}[W^\perp]$. In addition, I have shown that 'twised duality' holds in the Clifford antidual. Using this, I was able to provide an alternative proof of the Hilbert space case that bypasses the usual norm issues that arise from Hilbert space basis calculations.

Additional Research

Accompanying my main work, I have also worked with my advisor, Dr. Paul Robinson, on investigating some interesting consequences involving Cesàro convergent sequences, i.e. sequences which have convergent averages. Specifically, we are furthering work done by Halmos, who showed that if we take the usual sequential definition of continuity and replace it with Cesàro continuity, we get an interesting restriction on the functions available. Take the standard notion of sequential continuity: for a function f, if $a_n \to a \implies f(a_n) \to f(a)$, then f is continuous at a. If we replace these convergences with Cesàro convergence, then we find that this notion of Cesàro continuity forces f to be a linear polynomial. We have taken this notion one step further. In addition to examining the interplay between this Cesàro convergence and the standard notion of convergence, we have examined what happens when we adapt our notion of differentiability to involve Cesàro convergence. We replace the sequential notion of differentiability with the requirement that averages of the difference quotients converge. As it turns out, we have shown that this places a requirement on the functions just as before, except we found that these functions must be quadratic polynomials. The unusual quirk about this result is that in the sense of Halmos, quadratic polynomials are not continuous, but we have found that they are differentiable.

Future Research

My goal for future research, at the moment, is to answer the question of 'twisted duality' in the complex C^* Clifford algebra, in the complex von Neumann Clifford algebra, and Clifford Hilbert spaces and antidual in the context of complex subspaces. Also, since a Clifford algebra is largely determined by the form that is put on the vector space V, there is a rich source of potential research questions by varying the form and repeating the investigation. Indeed, we can consider different Clifford algebra structures and ask the question of 'twisted duality' once more. The form placed on a Clifford algebra is a symmetric bilinear form, but we could instead replace it with a symplectic form and consider notions of 'twisted duality' of the associated Weyl algebras.

Undergraduate Research Topics

In further addition to my main research, I have worked with my advisor on analyzing structures of simple relations of sets and the structure that can be observed. This particular material is very hands-on and can be easily accessible to students who are just entering their core mathematics degree requirement. Specifically, this material would show up in a beginning proof writing course, so a student can begin to be exposed to some simple research ideas that can then evolve with them as they move through their courses. Naturally, it is not expected that they carry this as their main focus but instead as a gateway into research ideas and how to think of new research questions.

Another idea that can be used would be questions relating to the Cesàro convergence notions mentioned before. There is a rich field of questions that can be asked by replacing our common notions of convergence with Cesàro convergence and asking if we obtain new theorems in these modified contexts. These types of questions would easily be available to students who have completed or are in the middle of their first course in advanced calculus. Since this is usually taken just prior to graduate work, this gives students another opportunity for early exposure to research before their graduate studies.